

The Mean of Multi-Object Trajectories

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Abstract—This paper introduces the concept of a mean for trajectories and multi-object trajectories (defined as sets or multi-sets of trajectories) along with algorithms for computing them. Specifically, we use the Fréchet mean, and metrics based on the optimal sub-pattern assignment (OSPA) construct, to extend the notion of average from vectors to trajectories and multi-object trajectories. Further, we develop efficient algorithms to compute these means using greedy search and Gibbs sampling. Using distributed multi-object tracking as an application, we demonstrate that the Fréchet mean approach to multi-object trajectory consensus significantly outperforms state-of-the-art distributed multi-object tracking methods.

Index Terms—mean, average, consensus, mean trajectory, mean multi-object trajectory, distributed multi-object fusion.

I. INTRODUCTION

Multi-object estimation involves determining the trajectories of an unknown, time-varying number of objects from noisy sensor measurements. Formally, a *trajectory* is a function that maps time indices from a discrete time window to elements in some state space, and a *multi-object trajectory* is a point pattern or a multi-set¹ of trajectories [1]. This general definition covers trajectories that are fragmented (i.e., objects can disappear and then reappear later), which is usually the case with the output of most tracking systems. Estimating the multi-object trajectory lies at the heart of many applications ranging from space exploration [2], surveillance [3], [4], computer vision [5], [6], to cell biology [7], [8].

A fundamental concept in estimation (and statistics) is the mean or average. Intuitively, the mean can be interpreted as the typical point and is, arguably, the most widely used summary statistic. In the presence of uncertainty, the mean is also useful as an estimator of the variable of interest. Additionally, averaging is the most popular form of consensus—reaching an agreement about an entity of interest—crucial for the fusion of data/information from multiple sources. Consensus has a wide range of applications in managerial science, statistics, engineering and computer science [9], [10], [11], [12]. In computer science and engineering, consensus-based algorithms are extensively used in distributed computing [13], energy systems [14], communication networks [15], remote sensing [16], sensor fusion [12], networked and distributed estimation [17], [18].

While it is customary that the mean is simply the sum of the data points divided by the number of data points, this is not the case in multi-object estimation. When the data points are

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¹A multi-set may contain repeated elements.

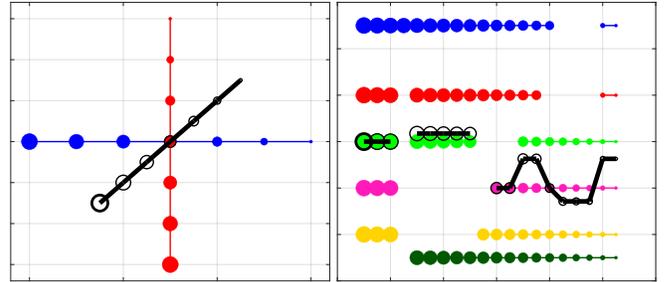


Figure 1. Two examples of the mean trajectory (thick black) of multiple input trajectories (color). Marker sizes indicate times, with the same size representing the same time, and sizes decrease toward the ends of trajectories. Consecutive instances of a trajectory are connected with solid lines.

trajectories of different lengths (see e.g., Figure 1), or multi-object trajectories (see e.g., Figure 2), the customary notion of average is not applicable because addition is not defined. The fundamental question then arises: what constitutes an “average” trajectory or an “average” multi-object trajectory? This question is crucial in multi-object estimation, yet, to the best of our knowledge, there is no existing work addressing the concept of an “average” trajectory or an “average” multi-object trajectory.

This work introduces the notion of average for trajectories and multi-object trajectories via the Fréchet mean, accompanied by algorithms for computing them. The Fréchet mean generalizes the concept of “average”, “centroid” or “barycenter” in Euclidean space to an arbitrary metric space [19]. In the same way as the Euclidean mean captures the geometric relationship induced by the Euclidean distance, the Fréchet mean captures the geometric relationship induced by the metric, allowing the physical interpretation as the “average” point. In a multi-object setting, the *optimal sub-pattern assignment* (OSPA) metric [20] is a meaningful distance between trajectories as well as between multi-sets/point patterns of trajectories [1]. OSPA-based distances and the concept of Fréchet mean enable the formulations of physically meaningful averages for trajectories and multi-object trajectories. In summary, the key contributions of this paper are:

- A formulation of the mean trajectory, and algorithms for computing this mean with respect to (w.r.t.) the OSPA trajectory metric;
- A formulation of the mean multi-object trajectory, and algorithms for computing this mean w.r.t. OSPA-based multi-object trajectory metrics.

Additionally, we demonstrate the capability of the proposed means in achieving consensus, using networked multi-object tracking as a specific area of application. Due to the proliferation of low-cost sensors that enable the deployment of large networks capable of monitoring large spatial regions, net-

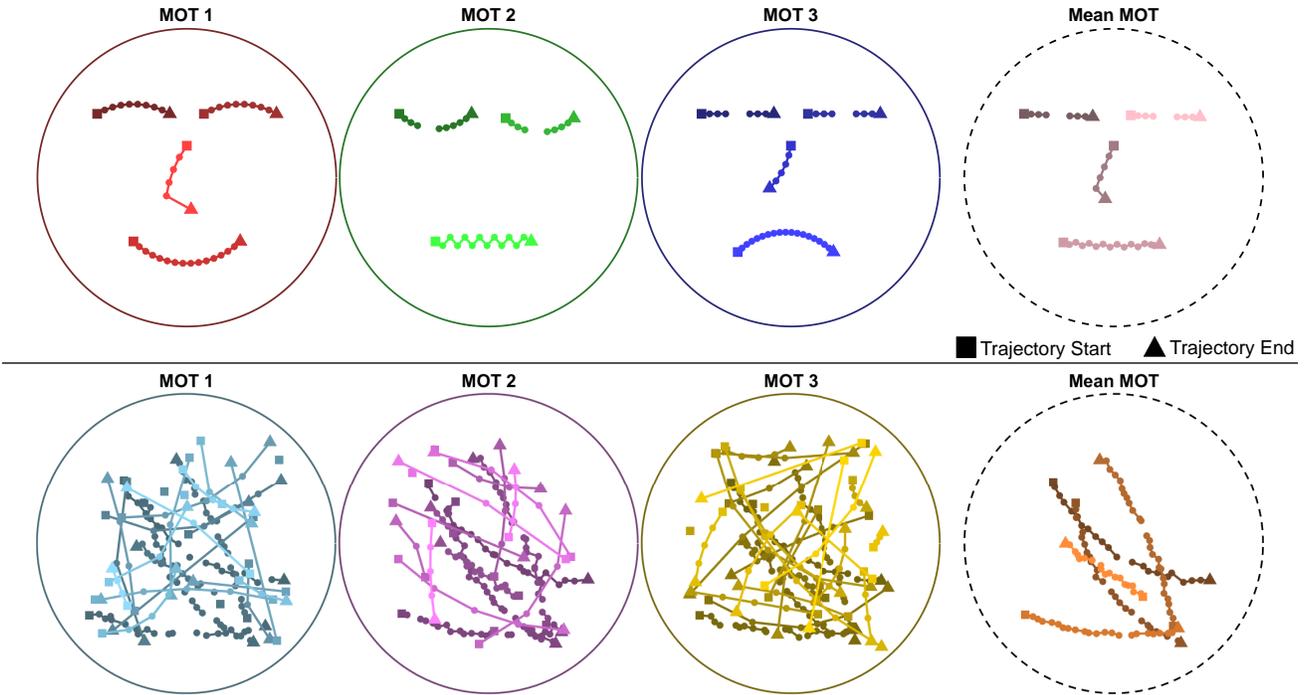


Figure 2. Examples of the mean multi-object trajectory (MOT) of three sample multi-object trajectories. Each trajectory in a multi-object trajectory has a distinct color. Consecutive instances of a trajectory are connected with solid lines. Intuitively, the mean tends to smooth out the data points.

worked multi-object tracking is an emerging trend [18]. This approach leverages advances in communication and sensing technologies to improve accuracy by fusing information from multiple nodes in a flexible and reliable manner, resilient to node failures [17].

Fusion is used to determine a multi-object trajectory description (e.g., estimates or probability distributions) that best agrees with the local multi-object trajectory descriptions at each node of the network. The fused description can be posed as a consensus on the local multi-object trajectories. The proposed Fréchet mean is a logical basis for consensus since by construction the mean is the closest (in aggregate distance) to all inputs from individual nodes. Further, results on the statistical consistency of Fréchet means have been established in [21]. Indeed, as illustrated in Figure 2, the Fréchet mean effectively suppresses noise and outliers. Notably, the proposed consensus can fuse node outputs regardless of the types of local trackers, and shows significant performance improvements over current state-of-the-art techniques.

The paper is organized as follows. Section II provides the background on averaging in multi-object trajectory consensus for distributed fusion, and metrics for multi-object states and multi-object trajectories. In Section III, we formulate the Fréchet mean trajectory w.r.t. the OSPA trajectory metric, and propose an efficient greedy algorithm to compute this trajectory mean. In Section IV we introduce the OSPA-based metrics for measuring distance between multi-object trajectories and the corresponding Fréchet mean, together with a greedy method to efficiently compute this mean. In Section V, we present methods to compute trajectory and multi-object trajectory means based on Gibbs sampling. Section VI demonstrates the performance of the proposed multi-object

trajectory consensus algorithm in a distributed multi-object tracking context. We conclude the paper and suggest potential extensions in Section VII.

II. BACKGROUND

This section discusses current techniques in distributed multi-object tracking pertinent to multi-object trajectory averaging, and metrics on multi-object states and trajectories that will be used to formulate the mean trajectory and mean multi-object trajectory.

A. Distributed Multi-Object Estimation

While there is no concept of mean or average multi-object trajectory, researchers in distributed multi-object tracking have developed fusion solutions based on indirect multi-object trajectory consensus. Assuming multi-object densities—functions on the class of finite sets of some (finite dimensional) space—are available, e.g., in the RFS approach, a popular consensus technique is to take the average of the relevant multi-object densities. The generalized covariance intersection (GCI) approach [22], also known as geometric averaging (GA), has been used routinely to fuse multi-object densities. GCI fusion algorithms for the probability hypothesis density (PHD) filter [23], the cardinalized PHD filter [24] and the multi-Bernoulli (MB) filter [25] have been proposed in [26], [27] and [28], respectively. Additionally, GCI-based fusion methods for the labeled multi-Bernoulli (LMB) filter [29] and the generalized LMB (GLMB) filter [30] have been proposed in [31], [32], [33], [34]. Alternatively, arithmetic averaging (AA) fusion has also been developed for multi-object density fusion, including AA-PHD [35], [36], AA-CPHD [37], AA-MB [38], and AA-LMB/GLMB [39]. The GA and AA fusion methods discussed

so far are intended for multi-object state fusion, and can be extended to multi-object trajectory fusion via the multi-scan multi-object density (i.e., joint density of the multi-object state over multiple scans). However, this approach to fusing multi-object trajectories is numerically infeasible due to the exponential growth in complexity.

An alternative to multi-scan multi-object density fusion is track-to-track (T2T) fusion [40]. This is an intuitive technique for fusing individual tracks together by minimizing some prescribed cost functions. Additional information on the state can be augmented into the cost function [41]. A popular cost function is based on the association probability of the sets of estimated objects with the physical objects [42], [43]. Alternatively, some authors use cost functions based on the statistical distance/divergence between the distributions describing the states of the tracks and their associated confidence [44], [45], [46], or the OSPA distance between the tracks [47]. T2T assignment can be cast as a multi-dimensional assignment problem, which is NP-hard for more than two dimensions. Nevertheless, sub-optimal solutions can be obtained using efficient clustering techniques, including hierarchical clustering [48], DBSCAN [49], density-peak clustering [50], or stochastic optimization [51]. In T2T fusion, clutter objects are typically not considered within the optimization process but are instead discarded based on existence probability [52], voting logic [53], or assignment confidence [54]. Determining an effective cost function for T2T fusion is still an active research topic [55], [56], [57].

Numerically, multi-object trajectory fusion is more efficient with T2T fusion since only the individual states or their distributions are needed, rather than the entire multi-object densities. In [58], the multi-scan assignment problem was solved using Lagrangian relaxation for a small number of sensors. A fast algorithm for multi-scan T2T assignment was proposed in [59], but it is limited to two sensors. The OSPA-based multi-object trajectory fusion approach proposed in [47] has been shown to be effective for a large number of sensors. Further, as a peer-to-peer fusion algorithm, it is more robust to network disruption. However, it is less accurate than global fusion algorithms, wherein all information is fused at once.

B. Metrics for Point Patterns and Trajectories

Since a multi-object state is a point pattern, metrics for point patterns are natural for measuring the distance between multi-object states. While the Hausdorff metric is an obvious point pattern metric, it is insensitive to cardinality differences [60]. To address this, the optimal mass transfer metric was proposed in [60], based on the Wasserstein construction. However, this metric suffers from a physical inconsistency if the cardinalities of the two multi-object states are not the same [60]. A widely used multi-object metric with meaningful physical interpretations, capturing both the state and cardinality errors, is the OSPA metric [20]. The OSPA construction (by this we mean the use of optimal sub-pattern assignment as the abbreviation suggests) also admits other metrics such as the cardinalized optimal linear assignment (COLA) metric [61]. The main difference is that the COLA is not normalized by the cardinality. Another unnormalized OSPA-based metric is a variant of the

COLA metric, misnamed as the Generalized OSPA (GOSPA) metric [62], which does not generalize the OSPA metric². Indeed, it is equivalent to the transport-transform (TT) metric proposed in [63], and thus, will be referred to as the TT metric since this is a more appropriate name. Apart from the mathematical similarity, it has been shown that COLA and TT metrics behave similarly in most scenarios, but TT is more sensitive to variations in parameters [64].

The TT and OSPA metrics are natural candidate distances to formulate the Fréchet mean of point patterns. Techniques for computing point pattern barycentres w.r.t. the TT and OSPA metrics have been discussed in [63]. An earlier approach for computing the OSPA barycenter using an alternating greedy search algorithm was proposed in [65]. This technique can be applied to approximate the minimum mean OSPA estimate (from certain distributions), first formulated in [66] for fixed cardinality, and later extended to unknown cardinality in [67].

In many applications, each element of the point pattern represents a probability distribution on \mathbb{R}^n to describe the associated uncertainty/confidence. Consequently, the discussed point pattern metrics need to be extended accordingly. Recall that the constructions of these metrics require a *base metric* or *base distance* between the elements. If the elements belong to \mathbb{R}^n , the p -norm of their difference is typically used as the base metric. When the elements are probability distributions, metrics such as Wasserstein [68], Hellinger, or geodesic distances on Gaussian manifolds [46], [69] can be used as the base metric. Indeed, the Hellinger metric was used as a base metric in [70] to incorporate estimation uncertainty into the OSPA metric. Further, information theoretic measures such as Kullback-Leibler's divergence (KLD) [45], Jeffrey's divergence (i.e., symmetrized KLD) [71], Bhattacharyya distance [44] can also be used to measure the dissimilarity between distributions, though they may not satisfy all metric properties.

In multi-object estimation, distances between multi-object trajectories are equally as important as distances between multi-object states. In [72] and [73], extensions of the OSPA metric (from multi-object states) to multi-object trajectories were proposed. However, these metrics are intractable, or are no longer proper metrics when numerically approximated in practice. A more practical approach is to treat a multi-object trajectory as a point pattern of trajectories, wherein point pattern metrics can be applied with tractable base distances between trajectories. A number of trajectory metrics were proposed in [74], but they exhibit counter-intuitive behavior, except for the OSPA-based construction. The OSPA metric with the OSPA-based trajectory base metric, called the OSPA⁽²⁾ metric, addresses the shortcomings of previous multi-object trajectory metrics, offering meaningful physical interpretation and can be efficiently computed for large-scale scenarios [1]. The TGOSPA metric [75], proposed after OSPA⁽²⁾, is not applicable in this work, because it was only proven for the special case of multi-object trajectories with contiguous trajectories. When the trajectories are sequences of probability distributions to describe the associated uncertainty, the OSPA-

²GOSPA does not subsume the OSPA metric as a special case. In fact GOSPA is still based on the OSPA construct.

Table I
LIST OF FREQUENTLY USED NOTATIONS.

Notation	Description
\mathbb{X}	state space
\mathbb{K}	finite time window $\{1, \dots, K\}$
\mathbb{T}	space of trajectories
$\mathcal{M}(\mathbb{T})$	space of multi-object trajectories
\mathbb{B}	$\{0, 1\}$
$1 : L$	$1, 2, \dots, L$
$\{1 : L\}$	$\{1, 2, \dots, L\}$
$x_{1:L}$	x_1, x_2, \dots, x_L
$x^{(1:L)}$	$x^{(1)}, x^{(2)}, \dots, x^{(L)}$
$d_{\mathbb{X}}^{(c)}(\cdot, \cdot)$	a base distance on \mathbb{X} with cut-off at c
$d_{\mathbb{T}}^{(c)}(\cdot, \cdot)$	a trajectory distance on \mathbb{T} with cut-off at c
\mathcal{D}_u	domain of trajectory u
$\bar{\mathcal{D}}_u$	complement of \mathcal{D}_u in \mathbb{K}
Π_L	the set of all permutation vectors on $\{1 : L\}$
$\mathbf{1}_X^{(x)}$ or $\mathbf{1}_X(x)$	equal 1 if $x \in X$, and equal 0 otherwise

based trajectory metric is constructed from distances between probability distributions similar to that for the OSPA metric for point patterns of distributions discussed above. Further details and discussions on metrics for multi-object states and multi-object trajectories can be found in [76].

III. TRAJECTORY AND FRÉCHET MEAN

This section presents a formulation and an accompanying numerical solution for trajectory consensus. Subsection III-A introduces the OSPA trajectory distance and formulates the Fréchet mean trajectory w.r.t. this distance. A method to compute the Fréchet mean trajectory based on greedy search is presented in Subsection III-B. A list of frequently used notation is given in Table I.

A. The Fréchet Mean Trajectory

Intuitively, the mean can be interpreted as the typical point of a data set. In a vector space, the arithmetic mean (i.e., the sum of the data points divided by the number of data points) minimizes the variance (i.e., the total squared Euclidean distance) from the data points. This property can be used to generalize the mean to an arbitrary metric space³ where addition is not meaningful [19]. More concisely, the notion of Fréchet mean is given as follows.

Definition 1. Let (d, \mathbb{M}) be a metric space and r be a positive integer. For $v^{(1)}, \dots, v^{(N)} \in \mathbb{M}$ and weights $w^{(1)}, \dots, w^{(N)} > 0$, the r^{th} -order weighted Fréchet mean is defined as

$$\hat{v} = \arg \min_{u \in \mathbb{M}} \sum_{n=1}^N w^{(n)} d^r(u, v^{(n)}). \quad (1)$$

When the weights are unity, \hat{v} is called the Fréchet mean.

Note that the Fréchet mean is not necessarily unique in general. The above sum can be interpreted as the Fréchet

³A space equipped with a metric for gauging the distance between any two elements, and a metric is a non-negative function of two variables, which satisfies the metric properties, see [1].

variance for $r = 2$. On a vector space equipped with the Euclidean distance, the above definition yields the arithmetic mean for $r = 2$, and the geometric median for $r = 1$.

A trajectory is a time-sequence of states, and can be either contiguous or fragmented [1], [77]. In a *contiguous* trajectory, the state exists at every instance from birth to death, whereas in a *fragmented* trajectory, there is at least an instance between birth and death that the state does not exist. Fragmentation manifests in the output of most tracking algorithms, mostly from incorrectly declaring that the object ceases to exist due to misdetections. The following definition covers both contiguous and fragmented trajectories [1].

Definition 2. A trajectory u , on a state space \mathbb{X} and a finite time window \mathbb{K} , is a function that maps \mathbb{K} to \mathbb{X} . The set of instances (in \mathbb{K}) where the state of trajectory u exists is the domain of u , denoted as \mathcal{D}_u . The space of all trajectories is denoted as \mathbb{T} .

Note that a trajectory u is contiguous if \mathcal{D}_u consists of consecutive instances, otherwise it is fragmented. The state space \mathbb{X} is usually a (finite dimensional) vector space or the space of probability distributions (on a vector space). The latter is used to accommodate uncertainty on the states of the trajectory at given instances.

Definition 3. Let $d_{\mathbb{X}}$ be a metric on the state space \mathbb{X} , and $d_{\mathbb{X}}^{(c)}(y, z) \triangleq \min\{d_{\mathbb{X}}(y, z), c\}$, $c > 0$. For any trajectories u and $v \in \mathbb{T}$, the OSPA trajectory distance of order $r \in [1, \infty]$ and cut-off c is defined as

$$d_{\mathbb{T}}^{(c,r)}(u, v) = \left[\frac{\sum_{k \in \mathcal{D}_u \cap \mathcal{D}_v} [d_{\mathbb{X}}^{(c)}(u(k), v(k))]^r + c^r (|\mathcal{D}_u - \mathcal{D}_v| + |\mathcal{D}_v - \mathcal{D}_u|)}{|\mathcal{D}_u \cup \mathcal{D}_v|} \right]^{\frac{1}{r}}. \quad (2)$$

Remark 1. The proof of the OSPA trajectory metric for $r = 1$ was given in [1]. Nonetheless, this result holds for any $r \in [1, \infty]$, and for completeness see Supplementary Materials (SM) Subsection I-A for proof. When the states of both trajectories exist (i.e., $k \in \mathcal{D}_u \cap \mathcal{D}_v$) the error is simply the distance between the respective states capped by c , and when only one state exists the error is c . Thus, intuitively the OSPA trajectory distance is simply the r^{th} -order time-averaged state and cardinality errors between two trajectories.

With respect to the metric $d_{\mathbb{T}}^{(c,r)}(\cdot, \cdot)$, the r^{th} -order Fréchet mean of the trajectories $v^{(1)}, \dots, v^{(N)} \in \mathbb{T}$ (hereon abbreviated as $v^{(1:N)}$) is

$$\hat{v} = \arg \min_{u \in \mathbb{T}} V^{(r)}(u), \quad (3)$$

where

$$V^{(r)}(u) = \sum_{k \in \mathbb{K}} \mathbf{1}_{\mathcal{D}_u}^{(k)} \Psi_k^{(r)}(u) + \mathbf{1}_{\bar{\mathcal{D}}_u}^{(k)} \bar{\Psi}_k^{(r)}(u), \quad (4)$$

$$\Psi_k^{(r)}(u) = \sum_{n=1}^N \frac{\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(k)} [d_{\mathbb{X}}^{(c)}(u(k), v^{(n)}(k))]^r + \mathbf{1}_{\bar{\mathcal{D}}_{v^{(n)}}}^{(k)} c^r}{|\mathcal{D}_u \cup \mathcal{D}_{v^{(n)}}|}, \quad (5)$$

$$\bar{\Psi}_k^{(r)}(u) = \sum_{n=1}^N \frac{\mathbf{1}_{\bar{\mathcal{D}}_{v^{(n)}}}^{(k)} c^r}{|\mathcal{D}_u \cup \mathcal{D}_{v^{(n)}}|}, \quad (6)$$

$\mathbf{1}_X^{(x)} = 1$ if $x \in X$ and 0 otherwise, and $\bar{\mathcal{D}}_u = \mathbb{K} - \mathcal{D}_u$. Note that the above functions implicitly depend on $v^{(1:N)}$. The expression for the cost function $V^{(r)}(u)$ is obtained by substituting (2) into the sum in (1) with unit weights.

Proposition 1. *Suppose \hat{v} is an r^{th} -order Fréchet mean of the trajectories $v^{(1:N)}$. Then for each $k \in \mathcal{D}_{\hat{v}}$, $\hat{v}(k)$ is the r^{th} -order Fréchet mean of the trajectory states $v^{(n)}(k)$ that exist at time k , weighted by $|\mathcal{D}_{\hat{v}} \cup \mathcal{D}_{v^{(n)}}|$, i.e.,*

$$\hat{v}(k) = \arg \min_{\mu \in \mathbb{X}} \sum_{n: \mathcal{D}_{v^{(n)}} \ni k} \frac{[d_{\mathbb{X}}^{(c)}(\mu, v^{(n)}(k))]^r}{|\mathcal{D}_{\hat{v}} \cup \mathcal{D}_{v^{(n)}}|}. \quad (7)$$

The above result (see SM Subsection I-B for proof) provides an explicit expression for the state of the mean trajectory at each time step, given its domain. In the next subsection, this result will be used to compute the mean trajectory.

B. Computing the OSPA Fréchet Mean Trajectory

This subsection presents a method to compute the Fréchet mean trajectory by converting Problem (3) into an equivalent form, and developing an efficient algorithm for solving it.

1) *Problem Transformation:* We transform the decision variables into an alternative representation on a different space by decomposing a trajectory u into its *existence history* γ and *states* x , via the transformation

$$\begin{aligned} \mathcal{A}: \mathbb{T} &\rightarrow \mathbb{B}^{|\mathbb{K}|} \times \mathbb{X}^{|\mathbb{K}|}, \\ u &\mapsto (\gamma, x), \end{aligned}$$

where $\mathbb{B} = \{0, 1\}$, while the k^{th} entries of $\gamma \in \mathbb{B}^{|\mathbb{K}|}$ and $x \in \mathbb{X}^{|\mathbb{K}|}$ are given by $\gamma_k = \mathbf{1}_{\mathcal{D}_u}^{(k)}$ and $x_k = \mathbf{1}_{\mathcal{D}_u}^{(k)} u(k)$, respectively (here we use the convention $0 \times u(k) = \mathbf{0} \in \mathbb{X}$ regardless of whether $u(k)$ is defined or not). If $\gamma_k = 1$, then the trajectory exists at the k^{th} instance and its state is given by x_k , otherwise, the trajectory does not exist at the k^{th} instance and x_k is ignored. For example, the 1D trajectory u , defined by $u(k) = k$ for each $k \in \{1, 2, 5\}$ on the window $\mathbb{K} = \{1 : 5\}$, can be represented by $\gamma = [1, 1, 0, 0, 1]$ and $x = [1, 2, 0, 0, 5]$.

A trajectory can be recovered from its existence history and states, by the transformation

$$\begin{aligned} \mathcal{R}: \mathbb{B}^{|\mathbb{K}|} \times \mathbb{X}^{|\mathbb{K}|} &\rightarrow \mathbb{T}, \\ (\gamma, x) &\mapsto u, \end{aligned}$$

where $u(k) = x_k$ whenever $\gamma_k = 1$, e.g., the 1D trajectory u with representation $\gamma = [1, 1, 0, 0, 1]$ and $x = [1, 2, 0, 0, 5]$, can be reconstructed by setting $\mathcal{D}_u = \{1, 2, 5\}$, $u(1) = x_1 = 1$, $u(2) = x_2 = 2$, and $u(5) = x_5 = 5$. It is clear that $\mathcal{R} \circ \mathcal{A} = \mathbf{I}$, where \mathbf{I} is the identity function, but \mathcal{R} is not the inverse of \mathcal{A} because there are more than one element of $\mathbb{B}^{|\mathbb{K}|} \times \mathbb{X}^{|\mathbb{K}|}$ that \mathcal{R} maps to the same u .

The following lemma provides the sufficient conditions for the equivalence of the original problem and its transformation.

Lemma 1. *Suppose that the transformations $\mathcal{A}: \mathbb{Z} \rightarrow \mathbb{Y}$ and $\mathcal{R}: \mathbb{Y} \rightarrow \mathbb{Z}$ satisfy $\mathcal{R} \circ \mathcal{A} = \mathbf{I}$ (\mathcal{A} is not necessarily invertible). Let $f: \mathbb{Z} \rightarrow [0, \infty)$ and $g: \mathbb{Y} \rightarrow [0, \infty)$ be such that*

$$g \circ \mathcal{A} = f \quad \text{and} \quad \mathcal{R}(y) = \mathcal{R}(y') \Rightarrow g(y) = g(y').$$

If y^ minimizes g , then $\mathcal{R}(y^*)$ minimizes f , and $g(y^*) = f(\mathcal{R}(y^*))$. (See SM Subsection I-C for proof)*

Using this equivalent representation, the Fréchet mean trajectory (3) is given by the solution to Problem (8) below on the space $\mathbb{B}^{|\mathbb{K}|} \times \mathbb{X}^{|\mathbb{K}|}$. This is stated more concisely in the following (see SM Subsection I-D for proof).

Proposition 2. *An r^{th} -order Fréchet mean of the trajectories $v^{(1:N)}$ is the trajectory \hat{v} , defined by $\hat{v}(k) = \hat{x}_k$ whenever $\hat{\gamma}_k = 1$, where*

$$(\hat{\gamma}, \hat{x}) = \arg \min_{(\gamma, x) \in \mathbb{B}^{|\mathbb{K}|} \times \mathbb{X}^{|\mathbb{K}|}} U^{(r)}(\gamma, x), \quad (8)$$

$$U^{(r)}(\gamma, x) = \sum_{k \in \mathbb{K}} \gamma_k \psi_k^{(r)}(\gamma, x_k) + (1 - \gamma_k) \bar{\psi}_k^{(r)}(\gamma), \quad (9)$$

$$\psi_k^{(r)}(\gamma, x_k) = \sum_{n=1}^N \frac{\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(k)} [d_{\mathbb{X}}^{(c)}(x_k, v^{(n)}(k))]^r + \mathbf{1}_{\bar{\mathcal{D}}_{v^{(n)}}}^{(k)} c^r}{|\mathcal{D}_{\gamma} \cup \mathcal{D}_{v^{(n)}}|}, \quad (10)$$

$$\bar{\psi}_k^{(r)}(\gamma) = \sum_{n=1}^N \frac{\mathbf{1}_{\bar{\mathcal{D}}_{v^{(n)}}}^{(k)} c^r}{|\mathcal{D}_{\gamma} \cup \mathcal{D}_{v^{(n)}}|}, \quad (11)$$

$$\mathcal{D}_{\gamma} = \{k \in \mathbb{K} : \gamma_k = 1\}. \quad (12)$$

Further, Problem (8) can be reduced to optimizing on $\mathbb{B}^{|\mathbb{K}|}$ by exploiting Proposition 1 as follows. For each $\gamma \in \mathbb{B}^{|\mathbb{K}|}$, let

$$x^*(\gamma) \triangleq \arg \min_{x \in \mathbb{X}^{|\mathbb{K}|}} U^{(r)}(\gamma, x), \quad (13)$$

$$W^{(r)}(\gamma) \triangleq U^{(r)}(\gamma, x^*(\gamma)). \quad (14)$$

It is clear that if γ^* minimizes $W^{(r)}$, then $(\gamma^*, x^*(\gamma^*))$ minimizes $U^{(r)}$, because $U^{(r)}(\gamma^*, x^*(\gamma^*)) = W^{(r)}(\gamma^*)$, and $W^{(r)}(\gamma^*) \leq U^{(r)}(\gamma, x^*(\gamma)) \leq U^{(r)}(\gamma, x)$. Using the property of the Fréchet mean in Proposition 1 we can compute $x^*(\gamma)$, and subsequently $W^{(r)}(\gamma)$ for any γ as summarized in Proposition 3 below. Hence, solving Problem (8) amounts to finding the minimizer γ^* of $W^{(r)}$, and then invoking Proposition 3 again to determine the states of the mean trajectory.

Proposition 3. *For any $\gamma \in \mathbb{B}^{|\mathbb{K}|}$, let $[x_1^*(\gamma), \dots, x_{|\mathbb{K}|}^*(\gamma)] \triangleq \arg \min_{x \in \mathbb{X}^{|\mathbb{K}|}} U^{(r)}(\gamma, x)$. If $\gamma_k = 0$ or $\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(k)} = 0$, for all $n = 1 : N$, then $x_k^*(\gamma) = \mathbf{0} \in \mathbb{X}$. Further, if $\gamma_k = 1$, then*

$$x_k^*(\gamma) = \arg \min_{x_k \in \mathbb{X}} \sum_{n: \mathcal{D}_{v^{(n)}} \ni k} \frac{[d_{\mathbb{X}}^{(c)}(x_k, v^{(n)}(k))]^r}{|\mathcal{D}_{\gamma} \cup \mathcal{D}_{v^{(n)}}|}. \quad (15)$$

This result (see SM Subsection I-E for proof) provides an explicit expression for each component of $x^*(\gamma)$. Specifically, if at time k , the state of $x^*(\gamma)$ does not exist ($\gamma_k = 0$) or all states of the sample trajectories do not exist, then $x_k^*(\gamma)$ is set to $\mathbf{0} \in \mathbb{X}$. If the state of $x^*(\gamma)$ exists at time k ($\gamma_k = 1$), then $x_k^*(\gamma)$ is given by the r^{th} -order Fréchet mean of the sample trajectory states that exist, weighted by $|\mathcal{D}_{\gamma} \cup \mathcal{D}_{v^{(n)}}|$. Note that $x_k^*(\gamma)$ in (15) may not be unique, but any minimizer of the cost would suffice.

2) *Mean Trajectory via Greedy Search:* Minimizing $W^{(r)}$ on $\mathbb{B}^{|\mathbb{K}|}$ is a discrete optimization problem that can be solved with non-linear binary programming or other stochastic optimization techniques such as simulated annealing or genetic

Algorithm 1 Greedy algorithm for trajectory consensus with T_t iterations and convergence threshold ϵ_t .

Output: Mean trajectory \hat{v} .

Initialize $\gamma^* := \mathbf{0}$;
 Set $\gamma_k^* := 1$ if $\sum_{n=1}^N \mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(k)} > 0$;
 $\text{cost}^* := W^{(r)}(\gamma^*)$; $\delta_{\text{cost}} := \infty$; $i := 0$;
while $i < T_t$ **and** $\delta_{\text{cost}} > \epsilon_t$ **do**
 $i++$;
 for $k := 1 : K$ **do** $\tilde{\gamma} := \gamma^*$; $\tilde{\gamma}_k := 1 - \gamma_k^*$;
 if $W^{(r)}(\tilde{\gamma}) < \text{cost}^*$ **then**
 $\gamma^* := \tilde{\gamma}$; $\delta_{\text{cost}} := \text{cost}^* - W^{(r)}(\tilde{\gamma})$; $\text{cost}^* := W^{(r)}(\tilde{\gamma})$;
 Compute x^* from γ^* using (15);
 Transform $\hat{v} := \mathcal{R}(\gamma^*, x^*)$;

algorithms [78]. Exhaustive search is also possible if the search space $\mathbb{B}^{|\mathbb{K}|}$ is small ($|\mathbb{K}|$ is small).

Targeting efficiency to suit online applications, we present a greedy search algorithm that generates a new $\tilde{\gamma}$ from γ simply by changing the component $\gamma_k \in \{0, 1\}$ to its complement (i.e., from 0 to 1 or vice-versa) if this reduces the cost $W^{(r)}$. This process is cycled through all values of $k \in \mathbb{K}$ until the cost reduction falls below some threshold. The pseudocode is given in Algorithm 1. If the complexity of solving Problem (15) is $\mathcal{O}(M)$, then the worst case complexity of this algorithm is $\mathcal{O}(T_t K^2 M)$. Figure 1 shows the mean trajectories computed using this method.

Observe from (9)-(13) that the dependence of trajectory $x^*(\gamma)$ on γ is captured in the factors $(|\mathcal{D}_\gamma \cup \mathcal{D}_{v^{(n)}}|)^{-1}$, $n \in \{1 : N\}$. In practice, these factors have similar values for different n since the estimates of the same trajectory, but by different nodes, would have similar lengths. Moreover, these lengths are relatively long, thereby diminishing the influence of $(|\mathcal{D}_\gamma \cup \mathcal{D}_{v^{(n)}}|)^{-1}$ on $x^*(\gamma)$. Hence, to further reduce computations, the optimal $x^*(\gamma)$ can be approximated by an x^* with components

$$x_k^* = \arg \min_{x_k \in \mathbb{X}} \sum_{n: \mathcal{D}_{v^{(n)}} \ni k} [d_{\mathbb{X}}^{(c)}(x_k, v_k^{(n)})]^r. \quad (16)$$

If $\{1 : N\} \cap \{n : \mathcal{D}_{v^{(n)}} \ni k\} = \emptyset$, we set $x_k = \mathbf{0} \in \mathbb{X}$. This approximation reduces the worst case complexity of Algorithm 1 to $\mathcal{O}(K(T_t + M))$. For the examples shown in Figure 1, this approximate cost produces almost identical mean trajectories to those from the exact cost.

IV. MULTI-OBJECT TRAJECTORY AND FRÉCHET MEAN

This section extends the notion of consensus for trajectories in the preceding section to multi-object trajectories. We start by formulating the Fréchet mean multi-object trajectory w.r.t. suitable OSPA-based metrics in Subsection IV-A, and then transform this problem to an equivalent form amenable to tractable solutions in Subsection IV-B.

A. The Fréchet Mean Multi-Object Trajectory

A multi-object trajectory \mathbf{X} is a point pattern⁴ of trajectories, i.e., each element of \mathbf{X} belongs to the space \mathbb{T} of

trajectories. Let $\mathcal{M}(\mathbb{T})$ denote the space of all multi-object trajectories. Then the r^{th} -order Fréchet mean of the multi-object trajectories $\mathbf{Y}^{(1:N)} = \mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(N)}$, w.r.t. a multi-object trajectory metric d , is given by

$$\hat{\mathbf{Y}} = \arg \min_{\mathbf{X} \in \mathcal{M}(\mathbb{T})} \sum_{n=1}^N d^r(\mathbf{X}, \mathbf{Y}^{(n)}). \quad (17)$$

Various multi-object trajectory metrics can be constructed using the OSPA-trajectory metric (presented in Section III-A) as the base metric, e.g., the Hausdorff-OSPA, Wasserstein-OSPA and OSPA-OSPA or OSPA⁽²⁾ metric [1], [76]. The Hausdorff is insensitive to cardinality differences [60] while the Wasserstein suffers from a physical inconsistency in cardinalities [60]. The OSPA⁽²⁾ metric, intuitively capturing the time-averaged cardinality and state errors per trajectory, alleviates these issues and is defined as follows.

Definition 4. Consider $r \geq 1$, $p \geq 0$, $c \in [p, \sqrt[p]{2p}]$, and the OSPA trajectory distance $d_{\mathbb{T}}^{(c,r)}$ on \mathbb{T} given by⁵ (2). For any two multi-object trajectories $\mathbf{X} = \{\mathbf{X}_{1:m}\}$ and $\mathbf{Y} = \{\mathbf{Y}_{1:s}\}$ (without loss of generality, assume $m \leq s$), the OSPA⁽²⁾ (multi-object trajectory) distance of order r , cardinality penalty p and cut-off c is defined as [63], [20], [1]

$$d(\mathbf{X}, \mathbf{Y}) = \left[\min_{\pi \in \Pi_s} \sum_{\ell=1}^m \frac{[d_{\mathbb{T}}^{(c,r)}(\mathbf{X}_\ell, \mathbf{Y}_{\pi(\ell)})]^r + |s-m|p^r}{s} \right]^{\frac{1}{r}}, \quad (18)$$

with $d(\emptyset, \emptyset) = 0$, where Π_s denotes the space of permutations on $\{1 : s\}$. If $m > s$, we define $d(\mathbf{X}, \mathbf{Y}) = d(\mathbf{Y}, \mathbf{X})$. Note that for simplicity, we suppressed the dependence on the parameters r , c and p . This definition is also based on the fact that the OSPA metric [20] can be extended to the case $c \in [p, \sqrt[p]{2p}]$, termed the relative TT metric in [63] (see Section 2 of the referred paper for more details).

Remark 2. If $s = m$, (18) is also the Wasserstein distance.

More generally $d(\mathbf{X}, \mathbf{Y})$ is the OSPA distance between the multi-object trajectories \mathbf{X}, \mathbf{Y} when $d_{\mathbb{T}}^{(c)}(y, z) = \min\{d_{\mathbb{T}}(y, z), c\}$, for any trajectory distance $d_{\mathbb{T}}(y, z)$. Note also that the OSPA construction admits variations of the OSPA metric such as the COLA metric [61], and the TT metric [63] by extending (18) to

$$d(\mathbf{X}, \mathbf{Y}) = \left[\min_{\pi \in \Pi_s} \sum_{\ell=1}^m \frac{[d_{\mathbb{T}}^{(c)}(\mathbf{X}_\ell, \mathbf{Y}_{\pi(\ell)})]^r + |s-m|p^r}{\kappa(s, m)} \right]^{\frac{1}{r}}, \quad (19)$$

where κ is a non-negative real function of non-negative integer pairs such that $\kappa(s, m) = \kappa(m, s)$. Setting: $\kappa(s, m) = \max(s, m)$ yields OSPA⁽²⁾; $\kappa(s, m) = c^r$, $p = c$ yields COLA-OSPA; and $\kappa(s, m) = 1$, $p = c/\sqrt[r]{\alpha}$ yields TT-OSPA.

An upper bound on the cardinality of the Fréchet mean multi-object trajectory w.r.t. the OSPA-based metrics is given in the following proposition. The proof is given in SM Subsection I-F.

⁵We can use $d_{\mathbb{T}}^{(c,r)}(y, z) = \min\{d_{\mathbb{T}}^{(q,l)}(y, z), c\}$, where $d_{\mathbb{T}}^{(q,l)}(y, z)$ is the OSPA trajectory distance, but for convenience we set $q = c$ and $l = r$.

⁴A point pattern is a multi-set and may contain repeated elements.

Proposition 4. Consider the Fréchet mean of $\mathbf{Y}^{(1:N)}$ w.r.t. the OSPA-based metric described by (19). If for each $j \geq L \triangleq \sum_{n=1}^N |\mathbf{Y}^{(n)}|$, $\kappa(j, |\mathbf{Y}^{(n)}|)$ depends only on j , i.e. $\kappa(j, |\mathbf{Y}^{(n)}|) = \kappa(j)$, and $\kappa(j+1) - \kappa(j) \leq \kappa(L)/L$. Then, the Fréchet mean's cardinality is no greater than L .

Note that OSPA⁽²⁾, COLA-OSPA and TT-OSPA all satisfy the premise of the above proposition. The next subsection presents a tractable method to compute the Fréchet mean in (17) w.r.t. an OSPA-based multi-object trajectory metric of the form (19).

B. Computing the Fréchet Mean Multi-Object Trajectory

Computing the mean multi-object trajectory requires solving an optimization problem on the space $\mathcal{M}(\mathbb{T})$, which is difficult since the decision variable involves both the cardinality and elements of the multi-object trajectory. In this subsection, we convert this problem into an equivalent form, and present an efficient algorithm for solving it.

1) *Problem Transformation:* Similar to Subsection III-B, we transform the decision variable in $\mathcal{M}(\mathbb{T})$ to an alternative representation on a different space by decomposing a multi-object trajectory $\mathbf{X} = \{\mathbf{X}_{1:m}\}$ into the *existence list* η and *trajectory list* τ , each with length $L \geq m$, via the transformation

$$\begin{aligned} \mathcal{A} : \mathcal{M}(\mathbb{T}) &\rightarrow \mathbb{B}^L \times \mathbb{T}^L, \\ \{\mathbf{X}_{1:m}\} &\mapsto (\eta, \tau), \end{aligned}$$

where the list entries $\eta_\ell = 1$, $\tau_\ell = \mathbf{X}_\ell$ for $\ell = 1 : m$, and $\eta_\ell = 0$, $\tau_\ell = \mathbf{0}$ (defined to be function that maps every element of \mathbb{K} to zero), otherwise. By convention, $\mathbf{X} = \{\}$ is represented by existence and trajectory lists of all zeros.

If $\eta_\ell = 1$, then the ℓ^{th} trajectory exists and takes on value τ_ℓ , otherwise the trajectory does not exist and whatever value of τ_ℓ is ignored. A multi-object trajectory can be recovered from its existence and trajectory lists by the transformation

$$\begin{aligned} \mathcal{R} : \mathbb{B}^L \times \mathbb{T}^L &\rightarrow \mathcal{M}(\mathbb{T}), \\ (\eta, \tau) &\mapsto \mathbf{X} = \{\tau_\ell : \eta_\ell = 1\}, \end{aligned}$$

It is clear that $\mathcal{R} \circ \mathcal{A} = \mathbf{I}$, but \mathcal{R} is not the inverse of \mathcal{A} because there are more than one element of $\mathbb{B}^L \times \mathbb{T}^L$ that \mathcal{R} maps to the same \mathbf{X} .

Since we are interested in computing the mean of the multi-object trajectories $\mathbf{Y}^{(1:N)}$, the list length L should not be less than the cardinality of this mean. Using the upper bound on the Fréchet mean in Proposition 4, we fix $L = \sum_{n=1}^N |\mathbf{Y}^{(n)}|$.

Note that Problem (17)-(19) requires computing an OSPA-based metric, which involves minimizing over the permutation space parameterized by the cardinality of the decision variable in $\mathcal{M}(\mathbb{T})$. Mapping the decision variable into $\mathbb{B}^L \times \mathbb{T}^L$ results in transforming this permutation space to the space of existence assignments defined as follows.

Definition 5. Given the existence lists η , $\xi^{(1:N)}$ in \mathbb{B}^L , an *existence assignment* between η and $\xi^{(n)}$ is a permutation π of $\{1 : L\}$ such that:

- $\eta_\ell = 1 \Rightarrow \xi_{\pi(\ell)}^{(n)} = 1$, if $\|\eta\|_1 \leq \|\xi^{(n)}\|_1$; or
- $\xi_{\pi(\ell)}^{(n)} = 1 \Rightarrow \eta_\ell = 1$, if $\|\eta\|_1 > \|\xi^{(n)}\|_1$.

We denote $\Omega^{(n)}(\eta)$ as the space of existence assignments between η and $\xi^{(n)}$, and $\Omega^{(1:N)}(\eta) \triangleq \Omega^{(1)}(\eta) \times \dots \times \Omega^{(N)}(\eta)$. Any element of $\Omega^{(n)}(\eta)$ can be represented as an L -vector $\omega^{(n)} = [\omega^{(n)}(\ell_1), \dots, \omega^{(n)}(\ell_L)]$, and any $\omega \in \Omega^{(1:N)}(\eta)$, can be represented as an $N \times L$ matrix with (n, ℓ) entry $\omega(n, \ell) = \omega^{(n)}(\ell)$.

Using the transformation \mathcal{A} above, Problem (17)-(19) on $\mathcal{M}(\mathbb{T})$ is equivalent to Problem (23) on $\mathbb{B}^L \times \mathbb{T}^L \times [\Pi_L]^N$. This is stated more concisely in the following (see SM Subsection I-G for proof).

Proposition 5. Let $\xi^{(1:N)}$, $\chi^{(1:N)}$ denote the respective existence and trajectory lists of the multi-object trajectories $\mathbf{Y}^{(1:N)}$, and define the functions

$$Q_\omega^{(r)}(\eta, \tau) = \sum_{\ell=1}^L \eta_\ell \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell) + (1 - \eta_\ell) \bar{\phi}_{\omega, \ell}^{(r)}(\eta), \quad (20)$$

$$\phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell) = \sum_{n=1}^N \frac{\xi_{\omega(n, \ell)}^{(n)} [d_{\mathbb{T}}^{(c)}(\tau_\ell, \chi_{\omega(n, \ell)}^{(n)})]^r + (1 - \xi_{\omega(n, \ell)}^{(n)}) p^r}{\kappa(\|\eta\|_1, \|\xi^{(n)}\|_1)}, \quad (21)$$

$$\bar{\phi}_{\omega, \ell}^{(r)}(\eta) = \sum_{n=1}^N \frac{\xi_{\omega(n, \ell)}^{(n)} p^r}{\kappa(\|\eta\|_1, \|\xi^{(n)}\|_1)}, \quad (22)$$

(which implicitly depend on $\xi^{(1:N)}$ and $\chi^{(1:N)}$). If

$$(\hat{\eta}, \hat{\tau}) = \arg \min_{(\eta, \tau) \in \mathbb{B}^L \times \mathbb{T}^L} \min_{\omega \in \Omega^{(1:N)}(\eta)} Q_\omega^{(r)}(\eta, \tau), \quad (23)$$

then the multi-object trajectory $\hat{\mathbf{X}} = \{\hat{\tau}_\ell : \hat{\eta}_\ell = 1\}$ is an r^{th} -order Fréchet mean of $\mathbf{Y}^{(1:N)}$.

Further, Problem (23) reduces to optimizing on $\mathbb{B}^L \times [\Pi_L]^N$ as follows. For each $\eta \in \mathbb{B}^L$ and $\omega \in [\Pi_L]^N$, let

$$\tau^*(\eta, \omega) \triangleq \arg \min_{\tau \in \mathbb{T}^L} Q_\omega^{(r)}(\eta, \tau), \quad (24)$$

$$S^{(r)}(\eta, \omega) \triangleq Q_\omega^{(r)}(\eta, \tau^*(\eta, \omega)), \quad (25)$$

and restricting ourselves to the constraint $\omega \in \Omega^{(1:N)}(\eta)$, i.e.,

$$\mathbf{1}_{\Omega^{(n)}(\eta_{1:L})}(\omega^{(n)}) = 1, n \in \{1 : N\}. \quad (26)$$

If (η^*, ω^*) minimizes $S^{(r)}$, subject to constraint (26), then $(\eta^*, \tau^*(\eta^*, \omega^*))$ minimizes $Q_{\omega^*}^{(r)}$, because $Q_{\omega^*}^{(r)}(\eta^*, \tau^*(\eta^*, \omega^*)) = S^{(r)}(\eta^*, \omega^*)$ and $S^{(r)}(\eta^*, \omega^*) \leq Q_{\omega^*}^{(r)}(\eta, \tau^*(\eta, \omega^*)) \leq Q_{\omega^*}^{(r)}(\eta, \tau)$. In addition, Proposition 6 below enables the computation of $\tau^*(\eta, \omega)$, and subsequently $S^{(r)}(\eta, \omega)$, for any feasible (η, ω) . Hence, Problem (23) translates to finding the minimizer (η^*, ω^*) of $S^{(r)}$, and invoking Proposition 6 again to determine the constituent trajectories of the optimal solution.

Proposition 6. For any $\eta \in \mathbb{B}^L$ and $\omega \in \Omega^{(1:N)}(\eta)$, let $[\tau_1^*(\eta, \omega), \dots, \tau_L^*(\eta, \omega)] = \arg \min_{\tau \in \mathbb{T}^L} Q_\omega^{(r)}(\eta, \tau)$. If $\eta_\ell = 0$, or $\xi_{\omega(n, \ell)}^{(n)} = 0$ for all $n = 1 : N$, then $\tau_\ell^*(\eta, \omega) = \mathbf{0}$. Otherwise,

$$\tau_\ell^*(\eta, \omega) = \arg \min_{\tau_\ell \in \mathbb{T}} \sum_{n: \xi_{\omega(n, \ell)}^{(n)} = 1} \frac{[d_{\mathbb{T}}^{(c)}(\tau_\ell, \chi_{\omega(n, \ell)}^{(n)})]^r}{\kappa(\|\eta\|_1, \|\xi^{(n)}\|_1)}. \quad (27)$$

Algorithm 2 Greedy algorithm for multi-object trajectory consensus with T_m iterations and convergence threshold ϵ_m .

Output: Mean multi-object trajectory \hat{X} .

Initialize $\eta := \mathbf{0}$, $\omega \in \Omega^{(1:N)}(\eta)$;
 ω , $\text{cost}^* := \text{optimizePermutation}(\eta, \omega)$;
 $\eta^* := \eta$; $\omega^* := \omega$; $\delta_{\text{cost}} := \infty$; $i := 0$;
while $i < T_m$ **and** $\delta_{\text{cost}} > \epsilon_m$ **do**
 $i++$;
 $\eta, \omega := \text{addTrajectories}(\eta, \omega)$;
 $\eta := \text{deleteTrajectories}(\eta, \omega)$;
 ω , $\text{cost} := \text{optimizePermutation}(\eta, \omega)$;
 if $\text{cost} < \text{cost}^*$ **then**
 $\eta^* := \eta$; $\omega^* := \omega$; $\delta_{\text{cost}} := \text{cost}^* - \text{cost}$; $\text{cost}^* := \text{cost}$;
 Compute τ^* from η^* and ω^* using (27);
 Transform $\hat{X} := \mathcal{R}(\eta^*, \tau^*)$;

Algorithm 3 optimizePermutation function.

Input: η , ω .
Output: Updated ω , cost .

Compute τ^* from η and ω using (27);
for $n := 1 : N$ **do**
 $\omega(n, \cdot) := \text{OptimalAssignment}((\eta, \tau^*), (\xi^{(n)}, \chi^{(n)}))$;
 $\text{cost} := S^{(r)}(\eta, \omega)$;

This result (see SM Subsection I-H for proof) provides an explicit expression for each component trajectory $\tau_\ell^*(\eta, \omega)$ of the trajectory list $\tau^*(\eta, \omega)$ that minimizes $Q_\omega^{(r)}(\eta, \cdot)$. Specifically, the ℓ^{th} trajectory $\tau_\ell^*(\eta, \omega)$ is set to $\mathbf{0} \in \mathbb{T}$, if $\eta_\ell = 0$ or $\tau_\ell^*(\eta, \omega)$ is not associated (via ω) with any sample trajectories. Otherwise, $\tau_\ell^*(\eta, \omega)$ is the r^{th} -order Fréchet mean of the associated sample trajectories $\chi_{\omega(1, \ell)}^{(1)}, \dots, \chi_{\omega(N, \ell)}^{(N)}$ that exist, weighted by $\kappa(\|\eta\|_1, \|\xi^{(1)}\|_1), \dots, \kappa(\|\eta\|_1, \|\xi^{(N)}\|_1)$, which can be computed using the method in Subsection III-B2. Note that the solution of Problem (27) might not be unique. Nonetheless, any minimizer of the cost would be sufficient.

2) *Mean Multi-Object Trajectory via Greedy Search:* Minimizing the cost function $S^{(r)}$ on $\mathbb{B}^L \times [\Pi_L]$ subject to constraint (26) is generally intractable. Inspired by the optimization method in [63], we present a greedy algorithm that alternates between η and ω until some local convergence condition is satisfied. This is outlined in Algorithm 2. First, we initialize $\eta = \mathbf{0}$ and any existence assignment matrix $\omega \in \Omega^{(1:N)}(\eta)$. We then alternate between minimizing: *i*) on the space $\Omega^{(1:N)}(\eta)$ to update ω ; and *ii*) on the space \mathbb{B}^L to update η .

For each $n \in \{1 : N\}$, the `optimizePermutation` function (Algorithm 3) is used to determine the optimal existence assignment $\omega^*(n, \cdot)$. In this step, we first determine the optimal trajectory list τ^* conditioned on η and the current ω by solving (27) for each $\ell \in \{1 : L\}$. To obtain $\omega^*(n, \cdot)$, we use a linear assignment solver [79] to assign elements of the set $\{\ell : \eta_\ell = 1\}$ to the set $\{\ell : \xi_\ell^{(n)} = 1\}$. The cost of assigning element i to j is $d_{\mathbb{T}}^{(c)}(\tau_i^*, \chi_j^{(n)})$. For any indices that remain unassigned or are not included in the optimal assignments, we randomly assign them since any one of these assignments yields the same cost.

Algorithm 4 addTrajectories function.

Input: η , ω .
Output: Updated η , ω .

Compute τ^* from η and ω using (27);
for $n := 1 : N$ **do**
 $\hat{U}^{(n)} := \{\chi_\ell^{(n)} \mid d_{\mathbb{T}}^{(c)}(\tau_\ell^*, \chi_{\omega(n, \ell)}^{(n)}) = c,$
 $\xi_{\omega(n, \ell)}^{(n)} = 1, \eta_\ell = 1, \ell = 1 : L\}$;
 $\bar{U}^{(n)} := \{\chi_\ell^{(n)} \mid \xi_{\omega(n, \ell)}^{(n)} = 1, \eta_\ell = 0, \ell = 1 : L\}$;
 $U^{(n)} := \hat{U}^{(n)} \cup \bar{U}^{(n)}$; $U := \cup_{n=1}^N U^{(n)}$;
 $L_A := \{\ell = 1 : L \mid \eta_\ell = 0\}$;
for $\ell \in L_A$ **do**
 if $U = \emptyset$ **then break**
 $u := \text{sampleOneElementUniformly}(U)$; $U := U - \{u\}$;
 $\hat{\omega} := \text{optimizeCluster}(u, \omega, \ell, U^{(1:N)})$;
 $\hat{\eta} := \eta$; $\hat{\eta}_\ell := 1$;
 if $S^{(r)}(\hat{\eta}, \hat{\omega}) < S^{(r)}(\eta, \omega)$ **then** $\eta := \hat{\eta}$; $\omega := \hat{\omega}$;

optimizeCluster sub-function

Input: u , ω , ℓ , $U^{(1:N)}$.
Output: $\hat{\omega}$.

$\hat{\omega} := \omega$;
for $n := 1 : N$ **do**
 $j := \arg \min_{j \in \{1 : |U^{(n)}|\}} d_{\mathbb{T}}^{(c)}(u, U_j^{(n)})$;
 $i := \text{originalIndex}(j)$; (index of $U_j^{(n)}$ in $\chi^{(n)}$)
 Find $\hat{\ell}$ such that $\hat{\omega}(n, \hat{\ell}) = i$;
 $\hat{i} := \hat{\omega}(n, \hat{\ell})$; $\hat{\omega}(n, \hat{\ell}) := i$; $\hat{\omega}(n, \hat{i}) := \hat{i}$;

Algorithm 5 deleteTrajectories function.

Input: η , ω .
Output: Updated η .

for $\ell := 1 : L$ **do**
 if $\eta_\ell = 1$ **do** $\hat{\eta} := \eta$; $\hat{\eta}_\ell := 0$;
 if $S^{(r)}(\hat{\eta}, \omega) < S^{(r)}(\eta, \omega)$ **then** $\eta := \hat{\eta}$;

The minimization on \mathbb{B}^L is carried out by optimizing each coordinate of η through the functions `addTrajectories` (Algorithm 4) and `deleteTrajectories` (Algorithm 5). The `addTrajectories` function updates each coordinate of η from 0 to 1 whenever such a change reduces the cost. The `deleteTrajectories` function updates each coordinate of η from 1 to 0 if doing so results in a cost reduction. If the complexity of the algorithm to solve Problem (27) is $\mathcal{O}(Z)$, the worst case complexity of Algorithm 2 is $\mathcal{O}(T_m L(NL^2 + Z))$. In Figure 2, we show examples of multi-object trajectory consensus of three point patterns computed by this algorithm.

V. FRÉCHET MEAN VIA GIBBS SAMPLING

While the greedy algorithms proposed in the previous subsections are inexpensive and well-suited for online applications, convergence to the global optima is an open question. In the absence of theoretical results, we endeavor to offer some empirical analysis by benchmarking the greedy solutions

against near-optimal solutions (in the context of trajectory and multi-object trajectory consensus). To this end, we propose a stochastic optimization technique that can reach optimality, albeit at the expense of increased computations.

Our proposed optimization method uses Gibbs sampling to draw samples from a distribution whose peaks align with the minima of the cost function, and retains samples with the least costs. Specifically, given a cost function $g : \mathbb{Y} \rightarrow [0, \infty)$ and a constraint function $F : \mathbb{Y} \rightarrow \{0, 1\}$, we define a distribution

$$\rho(y) \propto e^{-\alpha(g(y))} F(y), \quad (28)$$

on \mathbb{Y} , where α is some positive constant. Since $e^{-\alpha(\cdot)}$ is bounded, and monotonically decreasing on $[0, \infty)$, the maxima of ρ coincide with the minima of g subject to the constraint $F(y) = 1$ (this also holds if we replace $e^{-\alpha(\cdot)}$ by any bounded monotonically decreasing function). Hence, all samples from ρ satisfy the constraint $F(y) = 1$ and the most likely samples are the minima of g .

The Gibbs sampler generates a Markov chain, wherein a new iterate $\hat{y} = (\hat{y}_{1:K})$ is generated from the current iterate $y = (y_{1:K})$ by sampling in each coordinate from the corresponding conditional distribution $\rho_k(\hat{y}_k | \hat{y}_{1:k-1}, y_{k+1:K})$ as follows [80]

$$\begin{aligned} \hat{y}_1 &\sim \rho_1(\cdot | y_{2:K}) \\ \hat{y}_2 &\sim \rho_2(\cdot | \hat{y}_1, y_{3:K}) \\ &\vdots \\ \hat{y}_K &\sim \rho_K(\cdot | \hat{y}_{1:K-1}). \end{aligned}$$

Choosing the conditional distributions

$$\rho_k(\hat{y}_k | \hat{y}_{1:k-1}, y_{k+1:K}) \propto \rho(\hat{y}_{1:k}, y_{k+1:K}), \quad (29)$$

and assuming the chain can visit all possible states in a finite number of steps, it can be shown that the Markov chain converges to the stationary distribution ρ , i.e., when the chain runs sufficiently long, the subsequent iterates are distributed according to ρ . Moreover, starting from an initial $y \in \mathbb{Y}$, the Gibbs sampler converges to the distribution ρ at an exponential rate [81]. In Subsections V-A and V-B, we adapt the Gibbs sampler, respectively, to compute the mean trajectory and mean multi-object trajectory.

A. Gibbs Sampling for Mean Trajectory

In trajectory consensus, $\mathbb{Y} = \mathbb{B}^L$, $y = \gamma$, the cost function $g(y) = W^{(r)}(\gamma)$, and there is no constraint. For completeness, the pseudocode to compute the Fréchet mean trajectory is given in Algorithm 6, and convergence of the Gibbs sampler (to the stationary distribution at an exponential rate) directly follows from Lemma 4.3.4 of [81]. If the complexity of the algorithm to solve Problem (15) is $\mathcal{O}(M)$, the complexity of Algorithm 6 is $\mathcal{O}(G_t K^2 M)$.

Similar to the greedy algorithm introduced in Section III-B2, we can approximate the cost $W^{(r)}(\cdot)$ by using x^* computed by (16) in place of $x^*(\gamma)$. This approximation reduces the complexity of Algorithm 6 to $\mathcal{O}(K(G_t + M))$.

Algorithm 6 Gibbs sampling for trajectory consensus with G_t iterations and categorical sampler (denoted as `CatSampling`).

Output: Mean trajectory \hat{v} .

```

Initialize  $\gamma := \mathbf{0}$ ;
Set  $\gamma_k := 1$  if  $\sum_{n=1}^N \mathbf{1}_{D_v^{(n)}}^{(k)} > 0$ ;
 $\gamma^* := \gamma$ ;  $\text{cost}^* := W^{(r)}(\gamma^*)$ ;
for  $t := 1 : G_t$  do
  for  $k := 1 : K$  do
     $\gamma_k \sim \text{CatSampling}(\{0, 1\}, \rho_k(\cdot | \gamma_{1:k-1}, \gamma_{k+1:K}))$ ;
    if  $W^{(r)}(\gamma) < \text{cost}^*$  then  $\gamma^* := \gamma$ ;  $\text{cost}^* := W^{(r)}(\gamma)$ ;
  Compute  $x^*$  from  $\gamma^*$  using (15);
  Transform  $\hat{v} := \mathcal{R}(\gamma^*, x^*)$ ;

```

B. Gibbs Sampling for Mean Multi-Object Trajectory

In multi-object trajectory consensus, $\mathbb{Y} = \mathbb{B}^L \times [\Pi_L]^N$, $y_{1:L} = \eta_{1:L}$, $y_{L+1:K} = \omega^{(1:N)}$, $K = L + N$, the cost function $g(y_{1:K}) = S^{(r)}(\eta_{1:L}, \omega^{(1:N)})$ and the constraint function $F(y_{1:K}) = \prod_{n=1}^N \mathbf{1}_{\Omega^{(n)}(\eta_{1:L})}(\omega^{(n)})$, because each $\omega^{(n)}$ is an existence assignment, i.e., $\omega^{(n)} \in \Omega^{(n)}(\eta_{1:L})$. Hence, the stationary distribution is

$$\rho(\eta_{1:L}, \omega^{(1:N)}) \propto e^{-\alpha(S^{(r)}(\eta_{1:L}, \omega^{(1:N)}))} \prod_{n=1}^N \mathbf{1}_{\Omega^{(n)}(\eta_{1:L})}(\omega^{(n)}).$$

Since each $\omega^{(n)}$ is an existence assignment, i.e., a permutation vector in Π_L with certain properties, sampling $\hat{\omega}^{(n)}$ from the corresponding conditional

$$\rho_{L+n}(\cdot | \hat{\eta}_{1:L}, \hat{\omega}^{(1:n-1)}, \omega^{(n+1:N)}), \quad (30)$$

is more involved than sampling the existence variable $\eta_\ell \in \mathbb{B} = \{0, 1\}$. Sampling $\hat{\omega}^{(n)}$ coordinate by coordinate is difficult because each component of the permutation vector must be different from the others. Using block Gibbs sampling is expensive when the block size is large. To circumvent this issue, we propose a reparameterization technique based on *repeated insertion* [82] that enables efficient Gibbs sampling.

Given a *reference* permutation vector $\vartheta = [\vartheta_1, \dots, \vartheta_L]$, an *insertion vector* is a vector of positive integers $j = [j_1, \dots, j_L]$ that satisfies $j_\ell \leq \ell$, $\forall \ell \leq L$. A *repeated insertion function* ϱ_ϑ maps an insertion vector j to a permutation vector π , i.e., $\pi = \varrho_\vartheta(j)$, by inserting ϑ_ℓ into position j_ℓ of π and shifting its previous value (if there is one) to position $j_\ell + 1$, for each $\ell \leq L$. This is illustrated below.

Example 1. Consider a reference permutation vector $\vartheta = [3, 1, 2, 4]$ and an insertion vector $j = [1, 2, 2, 3]$. We start with an empty vector $\pi = [\circ, \circ, \circ, \circ]$. We first insert $\vartheta_1 = 3$ to position $j_1 = 1$ of π which yields $\pi = [3, \circ, \circ, \circ]$. We then insert $\vartheta_2 = 1$ to position $j_2 = 2$ of π which yields $\pi = [3, 1, \circ, \circ]$. We continue to insert $\vartheta_3 = 2$ to position $j_3 = 2$ of π . But since position 2 of π is currently occupied by 1, we shift 1 to position 3 and insert $\vartheta_3 = 2$ into position 2, which yields $\pi = [3, 2, 1, \circ]$. Finally we insert $\vartheta_4 = 4$ to position $j_4 = 3$. But since position 3 is currently occupied by 1, we shift 1 to position 4 and insert 4 in position 3, which finally yields $\pi = [3, 2, 4, 1]$.

Algorithm 7 Gibbs sampling for multi-object trajectory consensus with G_m iterations.

Output: Mean multi-object trajectory $\hat{\mathbf{X}}$.

Initialize η, ω using Algorithm 2;
 $\eta^* := \eta; \omega^* := \omega; \text{cost}^* := S^{(r)}(\eta, \omega);$
for $g := 1 : G_m$ **do**
 /*sample existence list*/
for $\ell := 1 : L$ **do**
 $\eta_\ell \sim \text{CatSampling}(\{0, 1\}, \rho_\ell(\cdot | \eta_{1:\ell-1}, \eta_{\ell+1:L}, \omega));$
 if $S^{(r)}(\eta, \omega) < \text{cost}^*$ **then**
 $\eta^* := \eta; \omega^* := \omega; \text{cost}^* := S^{(r)}(\eta, \omega);$
 /*sample existence assignments*/
for $n := 1 : N$ **do**
 $j := \varrho_\vartheta^{-1}(\omega^{(n)});$
 for $\ell := 1 : L$ **do**
 $j_\ell \sim \text{CatSampling}(\{1 : \ell\}, \tilde{p}_{\vartheta, \ell}(\cdot | j_{1:\ell-1}, j_{\ell+1:L}));$
 $\omega := [\omega^{(1)}, \dots, \omega^{(n-1)}, \varrho_\vartheta(j), \omega^{(n+1)}, \dots, \omega^{(N)}];$
 if $S^{(r)}(\eta, \omega) < \text{cost}^*$ **then**
 $\eta^* := \eta; \omega^* := \omega; \text{cost}^* := S^{(r)}(\eta, \omega);$
 Compute τ^* from η^* and ω^* using (27);
 Transform $\hat{\mathbf{X}} := \mathcal{R}(\eta^*, \tau^*);$

Noting that the repeated insertion function ϱ_ϑ is bijective [83], we can transform the problem of sampling a permutation vector π from a distribution $p(\cdot)$ (such as the conditional (29)), into sampling an insertion vector j from $\tilde{p}_\vartheta(\cdot) \triangleq p(\varrho_\vartheta(\cdot))$ and then recover $\pi = \varrho_\vartheta(j)$. Sampling insertion vectors from \tilde{p}_ϑ can be accomplished coordinate by coordinate via Gibbs sampling with conditionals

$$\tilde{p}_{\vartheta, \ell}(j_\ell | j_{1:\ell-1}, j_{\ell+1:L}) \propto \tilde{p}_\vartheta(j_{1:\ell-1}, j_\ell, j_{\ell+1:L}).$$

Specifically, for sampling $\omega^{(n)}$ from (30), the above conditional is proportional to

$$\rho(\eta_{1:L}, \omega^{(1:n-1)}, \varrho_\vartheta([j_{1:\ell}, j_{\ell+1:L}]), \omega^{(n+1:N)}).$$

The pseudocode to compute the mean multi-object trajectory via Gibbs sampling is provided in Algorithm 7, and convergence of the Gibbs sampler (to the stationary distribution at an exponential rate) is proven in SM Subsection II-A. If the complexity of the algorithm to solve Problem (27) is $\mathcal{O}(Z)$, the complexity of Algorithm 7 is $\mathcal{O}(G_m N L^3 Z)$, where G_m is the number of iterates of the Gibbs sampler. Implementation strategies to speed up computations are discussed in SM Subsection II-B.

Remark 3. Sampling distributions on the space of permutations is an important problem in computational statistics [84]. The proposed algorithm is the first to exploit repeated insertion to perform Gibbs sampling on the space of permutations.

VI. EXPERIMENTAL RESULTS

This section presents the first practical demonstration of Fréchet mean (FM) multi-object trajectory consensus, that is, the tractable computation of the geometric barycenter or “average” for multiple sets of trajectory valued elements. The results explore the behavior of FM consensus with the OSPA⁽²⁾

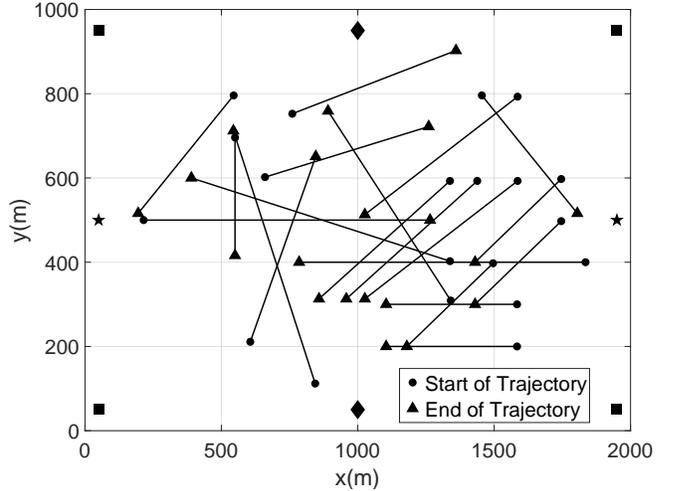


Figure 3. True trajectories and network configuration where the nodes are annotated by ■, ★ or ◆, see Table II for details of the sensor nodes.

Table II
SUMMARY OF SENSOR TYPES. THE PARAMETERS P_D AND λ_c ARE RESPECTIVELY THE SENSOR DETECTION PROBABILITY AND CLUTTER RATE.

Type	Symbol	Sensor	Tracker	P_D	λ_c
1	■	R-RR-B	TO-MHT	0.85	50
2	◆	POS	LMB	0.7	10
3	★	B-R	GLMB	0.9	70

metric (defined in (18)), where $p = 100\text{m}$ and $c = 100\sqrt{2}\text{m}$. We demonstrate the results for the first-order ($r = 1$) and the second-order ($r = 2$) Fréchet means, denoted as FM⁽¹⁾ and FM⁽²⁾ respectively. Both of the proposed implementations are considered, namely greedy search as in Sections III-IV (with 5 and 100 iterations respectively for trajectory and multi-object trajectory consensus) as well as Gibbs sampling as in Section IV (with 100 and 10 iterations respectively for trajectory and multi-object trajectory consensus).

Direct comparisons are also undertaken with the two latest approaches for trajectory consensus, DBSCAN clustering for tracks (DBSCAN-T) [49] and Track Consensus (TC) in [47]. It should be noted that the proposed FM consensus is inherently a multi-scan or window-based formulation (batch) whereas DBSCAN-T and TC are both single-scan or recursive formulations (scan-by-scan). Thus, the FM approach also marks the first tractable implementation of a batch as opposed to scan-by-scan algorithm for multi-object trajectory consensus.

A networked multi-object tracking scenario is employed to generate a synthetic dataset for the purpose of demonstrating multi-object trajectory consensus. A fixed ground truth with multiple sensor nodes is used to generate multiple sets of imperfect estimates. In general, any mix of sensor types or even human mediated reports can be processed since the FM or consensus is not dependent on the sensor types or models. The sets of node outputs are collectively fed into the consensus

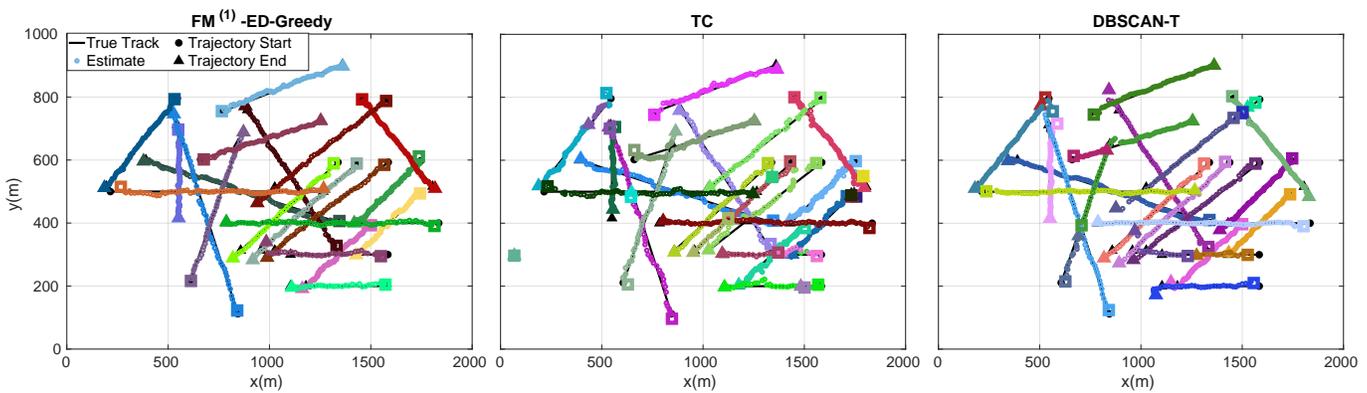


Figure 4. Sample outputs from $FM^{(1)}$ with Euclidean distance and greedy search, TC and DBSCAN-T.

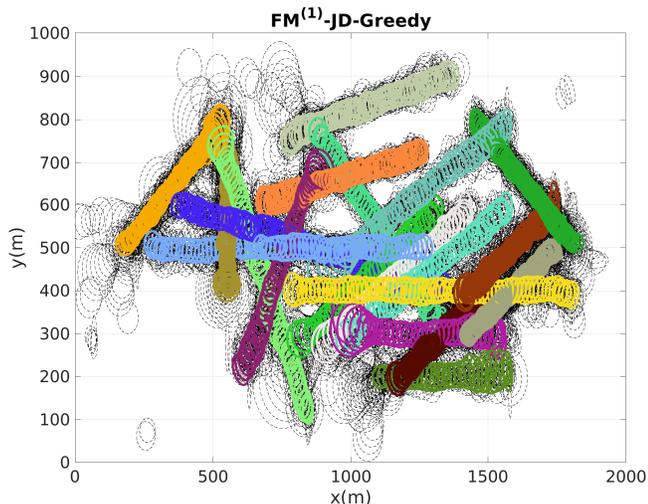


Figure 5. Sample output from $FM^{(1)}$ fusion with uncertainty. The ellipses represent 95% confidence intervals of the Gaussian distributions. Dashed line ellipses are estimates from the nodes and solid ellipses are the fused results.

algorithm whose result is then compared to the ground truth.

The ground truth has 20 trajectories of different lengths over 100 time scans of 1s intervals. The individual node reports come from 8 distinctly located sensors, each chosen from 3 different types, and producing estimates of the multi-object trajectory ground truth with varying accuracy. The ground truths and sensor locations are shown in Figure 3. The multi-object trajectory estimate from each node is simulated as the output of various tracking filters run on different measurement types. In particular, the measurement types are range, range-rate and bearing (R-RR-B), bearing range (B-R) and 2D position (POS) with noise standard deviation of 20m on position and range, 1° on bearing noise, and 2m/s on range rate. The trackers are track-oriented MHT (TO-MHT) [85], LMB [29], or GLMB filters [30]. All trackers use a constant velocity motion model with a sampling time of 1s and process noise standard deviation of $5m/s^2$. Node information is summarized in Table II.

In addition to deterministic estimates (wherein the Euclidean distance (ED) is used as the base distance), we also apply our technique to fuse multi-object trajectories with un-

certainty. Presumably, the reported trajectories from each node are represented by Gaussian distributions, i.e., if a trajectory exists at time k , its state is a Gaussian distribution parameterized by a mean vector and a covariance matrix. To illustrate the application of our technique in Fréchet-mean fusion of tracks with uncertainty, we use the techniques proposed in [71] to compute the centroids of Gaussians w.r.t. Jeffrey's divergence (JD)⁶. Since computing the centroid Gaussian distributions is expensive, we only compute the Fréchet mean multi-object trajectory (with uncertainty) using the greedy search method.

A sample output from each of $FM^{(1)}$ (Greedy) via ED, TC and DBSCAN-T is given in Figure 4. Further, we also illustrate the output from $FM^{(1)}$ fusion with uncertainty via JD in Figure 5. It can be seen that outputs of the FM approaches correctly identify all trajectories. In contrast, the TC output shows some susceptibility to track switching and fragmentation, while DBSCAN-T additionally exhibits late initiation and termination. These observations are consistent with the fact that $FM^{(1)}$ is a multi-scan or window-based formulation which processes batch data, whereas TC and DBSCAN-T are single-scan or recursive strategies which cannot change past estimates.

To confirm these observations, 100 Monte Carlo trials are performed. The instantaneous estimated cardinality versus time is shown in Figure 6. The OSPA⁽²⁾ metric [1], with order of 1, cut-off distance and cardinality penalty of 100m, is calculated on an expanding window and shown versus time in Figure 7. Importantly, the OSPA⁽²⁾ is a formal distance function which jointly penalizes localization and cardinality errors, in addition to trajectory fragmentation and labeling errors. The OSPA⁽²⁾ error value at a given instant can be interpreted as a time-averaged per-trajectory error, which in this study is calculated over an expanding window from the scenario start to the current time.

The cardinality curves in Figure 6 indicate that FM consensus generally results in correct cardinality estimates with quick track initiation but some delay in track termination. This behavior reflects the output of the underlying tracking filters at each of the sensor nodes since they all exhibit the

⁶JD, which is a symmetrized KLD, is a pseudo-metric since it does not satisfy the triangle inequality. We use JD in our demonstration owing to the popularity of KLD for fusing distributions via the geometric mean.

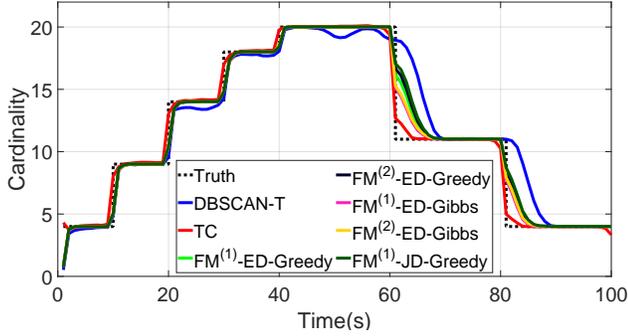
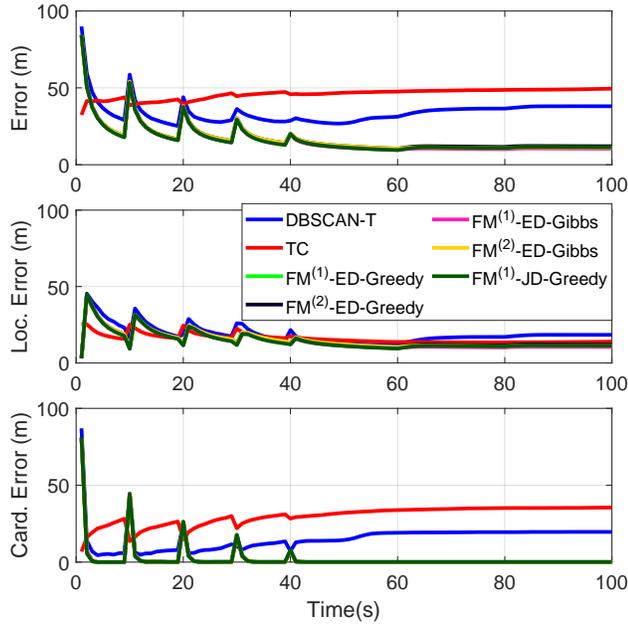


Figure 6. Average estimated cardinality.

Figure 7. Average OSPA⁽²⁾ over all MC runs at different time steps. All FM-based solutions have almost identical performance.

same characteristics which are reported as the consensus. The OSPA⁽²⁾ curves in Figure 7 further indicate that the greedy search (for both FM⁽¹⁾ and FM⁽²⁾), using either ED and JD as base distances, and Gibbs sampling implementations have almost identical performance. Importantly, the FM results in the lowest OSPA⁽²⁾ error of the three approaches since it implements window-based or multi-scan consensus scheme.

While TC has very accurate instantaneous cardinality estimates, the peer-to-peer nature exacerbates errors due to track switching and fragmentation, as indicated by the higher OSPA⁽²⁾ error. A similar examination of DBSCAN-T confirms the noticeable lag in track initialization and termination which occurs in addition to track switching and fragmentation. Even though DBSCAN-T is a centralized scheme, the single scan recursive nature combined with its generation of singleton clusters, results in more overall errors.

Further analysis is presented in Table III which shows the final OSPA⁽²⁾ error calculated over the entire scenario along with the total execution time. The final OSPA⁽²⁾ errors for

Table III
AVERAGE ERROR OVER ALL MC RUNS. ONE STANDARD DEVIATION SHOWN IN PARENTHESIS.

Method	OSPA ⁽²⁾ (m)	Exec. Time(s)
DBSCAN-T	38.05(5.38)	39.19(0.93)
TC	49.47(8.50)	397.36(7.24)
FM ⁽¹⁾ -ED (Greedy)	11.05(0.80)	27.57(2.29)
FM ⁽²⁾ -ED (Greedy)	12.05(1.05)	21.55(1.66)
FM ⁽¹⁾ -ED (Gibbs)	10.48(0.81)	1099.52(76.84)
FM ⁽²⁾ -ED (Gibbs)	11.21(0.81)	1073.81(61.44)
FM ⁽¹⁾ -JD (Greedy)	11.41(1.06)	154.47(17.59)

Table IV
AVERAGE ERROR OVER ALL MC RUNS FOR EACH NODE. ONE STANDARD DEVIATION SHOWN IN PARENTHESIS.

Node Loc.	OSPA ⁽²⁾ (m)	Node Loc.	OSPA ⁽²⁾ (m)
(50,950)	40.02(7.28)	(1950,50)	41.26(7.35)
(1000,950)	48.54(4.35)	(1000,50)	49.00(5.14)
(1950,950)	41.53(7.30)	(50,50)	39.24(6.87)
(1950,500)	61.56(4.35)	(50,500)	62.09(4.54)

each of the node estimates are also shown in Table IV. The final OSPA⁽²⁾ values confirm that FM consensus results in the lowest estimation error and significantly improves the raw sensor node outputs. Further, the greedy search implementation is marginally less accurate than Gibbs sampling but is orders of magnitude faster. Among the FM consensus algorithms, the second-order FM techniques exhibit slightly worse performance than their first-order counterparts, while demonstrating a noticeable reduction in run-time. The better performance of the order-1 FM could be attributed to the resilience of the median against noise. The more efficient order-2 FM (with the Euclidean base distance) is due to the simpler computation of the fused trajectory state at each time step by averaging the corresponding data points, whereas computing the median requires an iterative process. Overall, the results suggest that the FM via greedy search produces significant improvements in consensus performance compared to existing approaches and with relatively efficient computation times.

VII. CONCLUSIONS

In this work, we introduced the notion of average for trajectories and multi-sets of trajectories using the Fréchet mean and OSPA-based metrics. We proposed methods for computing these means based on greedy search or Gibbs sampling. The greedy-search-based algorithms are numerically efficient and reach consensus much faster than state-of-the-art algorithms. In the absence of theoretical results on the convergence to the optimal solution, we provide empirical analysis by benchmarking the greedy-based solution against the computationally intensive Gibbs-sampling-based solutions that have well established convergence properties. Our experiments on distributed multi-object tracking have shown that the proposed consensus solutions significantly outperform state-of-the-art techniques. The greedy search algorithms outperform the current techniques both in terms of efficiency and

accuracy by a significant margin, and are not far from the Gibbs-sampling-based solution in accuracy. Future research will focus on developing efficient algorithms for diverse tracking applications, including scenarios with unknown and non-common fields of view, where our consensus approach shows significant potential for adaptation and advancement.

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Supplementary Materials:

The Mean of Multi-Object Trajectories

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I. MATHEMATICAL PROOFS

In this section, we provide mathematical proofs that were left out in the main text.

A. OSPA Trajectory Metric Proof

Using the abbreviation $\mathcal{D}_{uv} = \mathcal{D}_u \cup \mathcal{D}_v$, we rewrite the OSPA distance (2) between any two trajectories $u, v \in \mathbb{T}$ as

$$d_{\mathbb{T}}^{(c,r)}(u,v) = \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_u \cup \mathcal{D}_v} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \right]^{\frac{1}{r}},$$

where $u_k = \{u(k)\}$ if $k \in \mathcal{D}_u$, $u_k = \emptyset$ if $k \notin \mathcal{D}_u$,

$$\tilde{d}_{\mathbb{X}}^{(c)}(x, y) = \begin{cases} d_{\mathbb{X}}^{(c)}(x^{(1)}, y^{(1)}), & |x| = |y| = 1 \\ 0, & |x| = |y| = 0 \\ c, & \text{otherwise} \end{cases}$$

and $x = \{x^{(1)}, \dots, x^{(n)}\}$. The proof is based on the following lemma.

Lemma A1. For any trajectories $u, v, t \in \mathbb{T}$, $c > 0$ and $r \in [1, \infty]$,

$$\left[\frac{1}{|\mathcal{D}_{ut}|} \sum_{k \in \mathcal{D}_{ut}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \right]^{\frac{1}{r}} \leq \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \right]^{\frac{1}{r}}.$$

Proof. For $r = \infty$, the inequality holds because

$$\lim_{r \rightarrow \infty} \left[\frac{1}{n} \sum_{i=1}^n [\alpha_i]^r \right]^{\frac{1}{r}} = \max_{1 \leq i \leq n} \alpha_i, \quad (\text{A1})$$

for $\alpha_i \geq 0$ and $\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k) = 0$ for $k \notin \mathcal{D}_{uv}$.

For $r \in [1, \infty)$, let $I = \mathcal{D}_{uv} \cap \mathcal{D}_{ut}$ and $V = \mathcal{D}_{uv} \setminus \mathcal{D}_{ut}$, which means $\mathcal{D}_{uv} = I \uplus V$. Noting that $|\mathcal{D}_{uv}| = |I| + |V|$ and $|I| = |\mathcal{D}_{uv} \cap \mathcal{D}_{ut}| \leq |\mathcal{D}_{ut}|$ we have,

$$\begin{aligned} & |\mathcal{D}_{uv}| |I| \leq |\mathcal{D}_{ut}| (|I| + |V|) \\ \Leftrightarrow & (|\mathcal{D}_{uv}| - |\mathcal{D}_{ut}|) |I| c^r \leq |\mathcal{D}_{ut}| |V| c^r \\ \Leftrightarrow & (|\mathcal{D}_{uv}| - |\mathcal{D}_{ut}|) \sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \leq |\mathcal{D}_{ut}| |V| c^r \\ \Leftrightarrow & |\mathcal{D}_{uv}| \sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \leq |\mathcal{D}_{ut}| \left[\sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r + |V| c^r \right] \\ \Leftrightarrow & \frac{1}{|\mathcal{D}_{ut}|} \sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \leq \frac{1}{|\mathcal{D}_{uv}|} \left[\sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r + |V| c^r \right]. \end{aligned}$$

Since $\mathcal{D}_{ut} = I \uplus (\mathcal{D}_{ut} \setminus \mathcal{D}_{uv})$ and $\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k) = 0$ for $k \notin \mathcal{D}_{uv}$,

$$\sum_{k \in \mathcal{D}_{ut}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r = \sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r.$$

In addition, since $\mathcal{D}_{uv} = I \uplus V$ and $[\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r = c^r$ for $k \in V$ (i.e., $k \in \mathcal{D}_v$, $k \notin \mathcal{D}_u$, $k \notin \mathcal{D}_t$),

$$\sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r = \sum_{k \in I} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r + |V| c^r.$$

Hence,

$$\frac{1}{|\mathcal{D}_{ut}|} \sum_{k \in \mathcal{D}_{ut}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \leq \frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r.$$

Raising both sides to $1/r$ yields the desired result. \blacksquare

By definition, (2) trivially satisfies the metric properties except for the triangle inequality. Thus, for any trajectories $u, v, t \in \mathbb{T}$, we need to prove

$$d_{\mathbb{T}}^{(c,r)}(u, v) \leq d_{\mathbb{T}}^{(c,r)}(v, t) + d_{\mathbb{T}}^{(c,r)}(u, t). \quad (\text{A2})$$

Consider $r \in [1, \infty)$. Since $\tilde{d}_{\mathbb{X}}^{(c)}$ is a metric,

$$\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k) \leq \tilde{d}_{\mathbb{X}}^{(c)}(v_k, t_k) + \tilde{d}_{\mathbb{X}}^{(c)}(u_k, t_k),$$

and hence,

$$\begin{aligned} & \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, v_k)]^r \right]^{\frac{1}{r}} \\ & \leq \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(v_k, t_k) + \tilde{d}_{\mathbb{X}}^{(c)}(u_k, t_k)]^r \right]^{\frac{1}{r}}. \end{aligned}$$

Using the definition of $d_{\mathbb{T}}^{(c,r)}$ and applying Minkowski's inequality to the right-hand side, it follows that

$$\begin{aligned} d_{\mathbb{T}}^{(c,r)}(u, v) & \leq \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(v_k, t_k)]^r \right]^{\frac{1}{r}} \\ & \quad + \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, t_k)]^r \right]^{\frac{1}{r}}. \end{aligned}$$

Applying Lemma A1 yields

$$\begin{aligned} & \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(v_k, t_k)]^r \right]^{\frac{1}{r}} \leq \left[\frac{1}{|\mathcal{D}_{vt}|} \sum_{k \in \mathcal{D}_{vt}} [\tilde{d}_{\mathbb{X}}^{(c)}(v_k, t_k)]^r \right]^{\frac{1}{r}}, \\ & \left[\frac{1}{|\mathcal{D}_{uv}|} \sum_{k \in \mathcal{D}_{uv}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, t_k)]^r \right]^{\frac{1}{r}} \leq \left[\frac{1}{|\mathcal{D}_{ut}|} \sum_{k \in \mathcal{D}_{ut}} [\tilde{d}_{\mathbb{X}}^{(c)}(u_k, t_k)]^r \right]^{\frac{1}{r}}. \end{aligned}$$

Thus, (A2) holds by the definition of $d_{\mathbb{T}}^{(c,r)}$ and the above inequality.

For $r = \infty$, it follows from (A1) that $\lim_{r \rightarrow \infty} d_{\mathbb{T}}^{(c,r)}(u, v) = d_{\mathbb{T}}^{(c,\infty)}(u, v)$. Thus, (A2) also holds using the same argument as per the $r \in [1, \infty)$ case.

B. Proof of Proposition 1

Suppose (7) is not true, i.e., there is an $l \in \mathcal{D}_{\hat{v}}$ and a $\mu \in \mathbb{X}$ such that

$$\sum_{n: \mathcal{D}_{v^{(n)}} \ni l} \frac{[d_{\mathbb{X}}^{(c)}(\mu, v^{(n)}(l))]^r}{|\mathcal{D}_{\hat{v}} \cup \mathcal{D}_{v^{(n)}}|} < \sum_{n: \mathcal{D}_{v^{(n)}} \ni l} \frac{[d_{\mathbb{X}}^{(c)}(\hat{v}(l), v^{(n)}(l))]^r}{|\mathcal{D}_{\hat{v}} \cup \mathcal{D}_{v^{(n)}}|}.$$

Let u be a trajectory with the same domain as \hat{v} , i.e., $\mathcal{D}_{\hat{v}} = \mathcal{D}_u$, where $u(l) = \mu$, and $u(k) = \hat{v}(k)$ for each $k \in \mathcal{D}_{\hat{v}} - \{l\}$. Then, adding $\sum_{n=1}^N \mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(l)} c^r |\mathcal{D}_u \cup \mathcal{D}_{v^{(n)}}|^{-1}$ to both sides of

the above inequality we have

$$\begin{aligned} \Psi_l^{(r)}(u) &= \sum_{n=1}^N \frac{\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(l)} [d_{\mathbb{X}}^{(c)}(\mu, v^{(n)}(l))]^r + \mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(l)} c^r}{|\mathcal{D}_u \cup \mathcal{D}_{v^{(n)}}|} \\ &< \sum_{n=1}^N \frac{\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(l)} [d_{\mathbb{X}}^{(c)}(\hat{v}(l), v^{(n)}(l))]^r + \mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(l)} c^r}{|\mathcal{D}_{\hat{v}} \cup \mathcal{D}_{v^{(n)}}|} = \Psi_l^{(r)}(\hat{v}). \end{aligned}$$

Further, $\mathbf{1}_{\mathcal{D}_{\hat{v}}}^{(k)} \bar{\Psi}_k^{(r)}(\hat{v}) = \mathbf{1}_{\mathcal{D}_u}^{(k)} \bar{\Psi}_k^{(r)}(u)$ because these terms only depend on $\mathcal{D}_{\hat{v}} = \mathcal{D}_u$. Hence,

$$\begin{aligned} V^{(r)}(u) &= \sum_{k \in \mathbb{K}} \mathbf{1}_{\mathcal{D}_u}^{(k)} \Psi_k^{(r)}(u) + \mathbf{1}_{\mathcal{D}_u}^{(k)} \bar{\Psi}_k^{(r)}(u) \\ &< \sum_{k \in \mathbb{K}} \mathbf{1}_{\mathcal{D}_{\hat{v}}}^{(k)} \Psi_k^{(r)}(\hat{v}) + \mathbf{1}_{\mathcal{D}_{\hat{v}}}^{(k)} \bar{\Psi}_k^{(r)}(\hat{v}) = V^{(r)}(\hat{v}). \end{aligned}$$

This contradicts the minimality of \hat{v} , hence the result.

C. Proof of Lemma 1

Let $y^* \in \mathbb{Y}$ be a minimizer of g , and $z^* = \mathcal{R}(y^*) \in \mathbb{Z}$. Since $\mathcal{R} \circ \mathcal{A} = \mathbf{I}$, we have $\mathcal{R}(\mathcal{A}(z^*)) = z^* = \mathcal{R}(y^*)$, which implies $g(\mathcal{A}(z^*)) = g(y^*)$. Further, using $g \circ \mathcal{A} = f$, we have $f(z^*) = g(\mathcal{A}(z^*)) = g(y^*)$.

Suppose there exists a $\hat{z} \in \mathbb{Z}$ such that $f(\hat{z}) < f(z^*)$. Then $g(\mathcal{A}(\hat{z})) = f(\hat{z}) < f(z^*) = g(y^*)$, which contradicts the minimality of y^* . Hence, if y^* minimizes g , then $z^* = \mathcal{R}(y^*)$ minimizes f , and $f(z^*) = g(y^*)$.

D. Proof of Proposition 2

It can be verified that $U^{(r)} \circ \mathcal{A} = V^{(r)}$, by substituting $(\gamma, x) = \mathcal{A}(u)$, i.e., $\gamma_k = \mathbf{1}_{\mathcal{D}_u}^{(k)}$, $x_k = \mathbf{1}_{\mathcal{D}_u}^{(k)} u(k)$, and $\mathcal{D}_\gamma = \mathcal{D}_u$ into $U^{(r)}(\gamma, x)$. Additionally, if $\mathcal{R}(\gamma, x) = \mathcal{R}(\gamma', x')$, then $\gamma = \gamma'$ and $x_k = x'_k$ whenever $\gamma_k = 1$, which implies $\gamma_k \psi_k^{(r)}(\gamma, x_k) = \gamma'_k \psi_k^{(r)}(\gamma', x'_k)$, and hence, $U^{(r)}(\gamma, x) = U^{(r)}(\gamma', x')$. Since $\mathcal{R} \circ \mathcal{A} = \mathbf{I}$, by virtue of Lemma 1, solving (3) amounts to solving (8) and applying the transformation \mathcal{R} to recover the optimal trajectory.

E. Proof of Proposition 3

Using the definition of $U^{(r)}(\gamma, x)$ in (9),

$$\begin{aligned} U^{(r)}(\gamma, x^*(\gamma)) &= \min_{x \in \mathbb{X}^{|\mathbb{K}|}} \sum_{k \in \mathbb{K}} \gamma_k \psi_k^{(r)}(\gamma, x_k) + (1 - \gamma_k) \bar{\psi}_k^{(r)}(\gamma) \\ &= \sum_{k \in \mathbb{K}} \min_{x_k \in \mathbb{X}} \gamma_k \psi_k^{(r)}(\gamma, x_k) + \sum_{k \in \mathbb{K}} (1 - \gamma_k) \bar{\psi}_k^{(r)}(\gamma). \end{aligned}$$

Hence, for each k , $x_k^*(\gamma) = \arg \min_{x_k \in \mathbb{X}} \gamma_k \psi_k^{(r)}(\gamma, x_k)$. If $\gamma_k = 0$ or $\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(k)} = 0$, for all $n = 1 : N$, then any value of $x_k^*(\gamma) \in \mathbb{X}$ yields the same cost. Thus, $x_k^*(\gamma) = \mathbf{0} \in \mathbb{X}$ is an optimal solution. If $\gamma_k = 1$, then $\gamma_k \psi_k^{(r)}(\gamma, \cdot) = \psi_k^{(r)}(\gamma, \cdot)$, and

$$\begin{aligned} x_k^*(\gamma) &= \arg \min_{x_k \in \mathbb{X}} \psi_k^{(r)}(\gamma, x_k) \\ &= \arg \min_{x_k \in \mathbb{X}} \sum_{n=1}^N \frac{\mathbf{1}_{\mathcal{D}_{v^{(n)}}}^{(k)} [d_{\mathbb{X}}^{(c)}(x_k, v^{(n)}(k))]^r}{|\mathcal{D}_\gamma \cup \mathcal{D}_{v^{(n)}}|}. \end{aligned}$$

F. Proof of Proposition 4

If $L \triangleq \sum_{n=1}^N |\mathbf{Y}^{(n)}| = 0$, an optimal solution (of Problem (17)-(19)) is $\hat{\mathbf{Y}} = \emptyset$, yielding zero cost.

We now consider $L > 0$. For $p = 0$, the cardinality difference has no contribution to the cost and an optimal solution is $\hat{\mathbf{Y}} = \cup_{n=1}^N \mathbf{Y}^{(n)}$, yielding zero cost, hence the bound holds. For $p > 0$, note that for each $\mathbf{Y}^{(i)}$, the multi-object trajectory $\bar{\mathbf{Y}} \triangleq \uplus_{j=1}^N \mathbf{Y}^{(j)}$ matches every elements of $\mathbf{Y}^{(i)}$, and the remaining unmatched elements from $\mathbf{Y}^{(1:i-1)}$, $\mathbf{Y}^{(i+1:N)}$ constitute a cost of

$$\frac{p^r \sum_{j=1, j \neq i}^N |\mathbf{Y}^{(j)}|}{\kappa(\sum_{i=1}^N |\mathbf{Y}^{(i)}|)}.$$

Hence, the total cost of $\bar{\mathbf{Y}}$ is

$$\begin{aligned} \frac{\sum_{i=1}^N \left(p^r \sum_{j=1, j \neq i}^N |\mathbf{Y}^{(j)}| \right)}{\kappa(\sum_{i=1}^N |\mathbf{Y}^{(i)}|)} &= \frac{(N-1)p^r \sum_{i=1}^N |\mathbf{Y}^{(i)}|}{\kappa(\sum_{i=1}^N |\mathbf{Y}^{(i)}|)}, \\ &= \frac{(N-1)p^r L}{\kappa(L)}. \end{aligned}$$

Now, suppose $\hat{\mathbf{Y}}$ is an optimal solution with cost D , and cardinality $|\hat{\mathbf{Y}}| > L$. Then, the optimality of $\hat{\mathbf{Y}}$ implies $D \leq \frac{(N-1)p^r L}{\kappa(L)}$, i.e.,

$$D \frac{\kappa(L)}{L} \leq (N-1)p^r,$$

and there exists (at least) a trajectory $u \in \hat{\mathbf{Y}}$ that is not paired to any sample trajectory in $\mathbf{Y}^{(1:N)}$. Removing u from $\hat{\mathbf{Y}}$ changes the cost by

$$\begin{aligned} \delta &= \frac{D\kappa(|\hat{\mathbf{Y}}|) - Np^r}{\kappa(|\hat{\mathbf{Y}}| - 1)} - D \\ &= \frac{D \left[\kappa(|\hat{\mathbf{Y}}|) - \kappa(|\hat{\mathbf{Y}}| - 1) \right] - (N-1)p^r - p^r}{\kappa(|\hat{\mathbf{Y}}| - 1)}. \end{aligned}$$

Since $|\hat{\mathbf{Y}}| > L$ and $\kappa(j+1) - \kappa(j) \leq \frac{\kappa(L)}{L}$ for all $j \geq L$,

$$D \left[\kappa(|\hat{\mathbf{Y}}|) - \kappa(|\hat{\mathbf{Y}}| - 1) \right] \leq D \frac{\kappa(L)}{L} \leq (N-1)p^r,$$

which means $\delta < 0$. Thus, there is a multi-object trajectory of cardinality $|\hat{\mathbf{Y}}| - 1$ with smaller cost than $\hat{\mathbf{Y}}$, thereby contradicting the optimality of $\hat{\mathbf{Y}}$. Hence, it is not possible to have an optimal solution with cardinality greater than L .

G. Proof of Proposition 5

Denote $[a, b] = \max\{a, b\}$, $[a, b] = \min\{a, b\}$, substituting (19) into (17) without assuming the larger cardinality yields

$$\begin{aligned} \hat{\mathbf{Y}} &= \arg \min_{\mathbf{X} \in \mathcal{M}(\mathbb{T})} \sum_{n=1}^N \min_{\pi^{(n)} \in \prod_{|\mathbf{x}|, |\mathbf{y}^{(n)}|} } \\ &\quad \sum_{\ell=1}^{|\mathbf{X}|, |\mathbf{Y}^{(n)}|} \frac{[d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y}^{(n)} | \ell, \pi^{(n)})]^r}{\kappa(|\mathbf{X}|, |\mathbf{Y}^{(n)}|)} + \frac{||\mathbf{Y}^{(n)}| - |\mathbf{X}|| p^r}{\kappa(|\mathbf{X}|, |\mathbf{Y}^{(n)}|)}, \end{aligned}$$

where

$$d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y} | \ell, \pi) = \begin{cases} d_{\mathbb{T}}^{(c)}(\mathbf{X}_\ell, \mathbf{Y}_{\pi(\ell)}), & |\mathbf{X}| \leq |\mathbf{Y}| \\ d_{\mathbb{T}}^{(c)}(\mathbf{X}_{\pi(\ell)}, \mathbf{Y}_\ell), & |\mathbf{X}| > |\mathbf{Y}| \end{cases}.$$

For some $L \geq \lceil |\mathbf{X}|, |\mathbf{Y}^{(n)}| \rceil$, we define $\varepsilon : \mathcal{M}(\mathbb{T}) \rightarrow \mathbb{B}^L$ such that for a multi-object trajectory \mathbf{X} , $\varepsilon_\ell(\mathbf{X}) = 1$ for $\ell \leq |\mathbf{X}|$ and $\varepsilon_\ell(\mathbf{X}) = 0$ for $\ell > |\mathbf{X}|$, $\varepsilon_\ell(\mathbf{X})$ is the ℓ^{th} component of $\varepsilon(\mathbf{X})$ and $\varepsilon(\mathbf{X})$ is the existence list of \mathbf{X} . We also define a space of permutation $\tilde{\Pi}_L^{(n)}(\varepsilon(\mathbf{X})) \subseteq \Pi_L$ such that for each $\tilde{\pi} \in \tilde{\Pi}_L^{(n)}(\varepsilon(\mathbf{X}))$

- $\varepsilon_\ell(\mathbf{X}) = 1 \Rightarrow \varepsilon_{\tilde{\pi}(\ell)}(\mathbf{Y}^{(n)}) = 1$, if $\|\varepsilon(\mathbf{X})\|_1 \leq \|\varepsilon(\mathbf{Y}^{(n)})\|_1$; or
- $\varepsilon_{\tilde{\pi}(\ell)}(\mathbf{Y}^{(n)}) = 1 \Rightarrow \varepsilon_\ell(\mathbf{X}) = 1$, if $\|\varepsilon(\mathbf{X})\|_1 > \|\varepsilon(\mathbf{Y}^{(n)})\|_1$.

We obtain the equivalence

$$\begin{aligned} & \min_{\pi \in \Pi_{\lceil |\mathbf{X}|, |\mathbf{Y}^{(n)}| \rceil}} \sum_{\ell=1}^L [d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y}^{(n)} | \ell, \pi)]^r \\ &= \min_{\tilde{\pi} \in \tilde{\Pi}_L^{(n)}(\varepsilon(\mathbf{X}))} \sum_{\ell=1}^L [d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y}^{(n)} | \ell, \tilde{\pi})]^r, \text{ and} \end{aligned}$$

$$\|\mathbf{Y}^{(n)}\| - |\mathbf{X}| = \sum_{\ell=1}^L \varepsilon_\ell(\mathbf{X})(1 - \varepsilon_\ell(\mathbf{Y}^{(n)})) + (1 - \varepsilon_\ell(\mathbf{X}))\varepsilon_\ell(\mathbf{Y}^{(n)}),$$

where

$$d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y} | \ell, \tilde{\pi}) = \begin{cases} d_{\mathbb{T}}^{(c)}(\mathbf{X}_\ell, \mathbf{Y}_{\tilde{\pi}(\ell)}), & \ell \leq |\mathbf{X}|, \tilde{\pi}(\ell) \leq |\mathbf{Y}| \\ 0, & \text{otherwise} \end{cases}$$

Further, if we fix $L = \sum_{n=1}^N |\mathbf{Y}^{(n)}|$ then $\tilde{\Pi}_L^{(n)}(\varepsilon(\mathbf{X})) = \Omega^{(n)}(\varepsilon(\mathbf{X}))$. Recall $(\xi^{(n)}, \chi^{(n)}) = \mathcal{A}(\mathbf{Y}^{(n)})$ and note that $\varepsilon(\mathbf{Y}^{(n)}) = \xi^{(n)}$, the original optimization problem becomes

$$\hat{\mathbf{Y}} = \arg \min_{\mathbf{X} \in \mathcal{M}(\mathbb{T})} \min_{\omega \in \Omega^{(1:N)}(\varepsilon(\mathbf{X}))} P_\omega^{(r)}(\mathbf{X}),$$

where

$$\begin{aligned} P_\omega^{(r)}(\mathbf{X}) &= \sum_{n=1}^N \sum_{\ell=1}^L \frac{[d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y}^{(n)} | \ell, \omega(n, \cdot))]^r}{\kappa(|\mathbf{X}|, |\mathbf{Y}^{(n)}|)} \\ &+ \frac{\varepsilon_\ell(\mathbf{X})(1 - \xi_{\omega(n, \ell)}^{(n)})p^r}{\kappa(|\mathbf{X}|, |\mathbf{Y}^{(n)}|)} + \frac{(1 - \varepsilon_\ell(\mathbf{X}))\xi_{\omega(n, \ell)}^{(n)}p^r}{\kappa(|\mathbf{X}|, |\mathbf{Y}^{(n)}|)}. \end{aligned}$$

Let $Q^{(r)}(\eta, \tau) = \min_{\omega \in \Omega^{(1:N)}(\eta)} Q_\omega^{(r)}(\eta, \tau)$ and $P^{(r)}(\mathbf{X}) = \min_{\omega \in \Omega^{(1:N)}(\varepsilon(\mathbf{X}))} P_\omega^{(r)}(\mathbf{X})$. It can be verified that $Q^{(r)} \circ \mathcal{A} = P^{(r)}$, by substituting $(\eta, \tau) = \mathcal{A}(\mathbf{X})$ into the expression of $P^{(r)}$, i.e., $\varepsilon(\mathbf{X}) = \eta$, $|\mathbf{X}| = \|\eta\|_1$, $\mathbf{X}_\ell = \tau_\ell$, and $d_{\mathbb{T}}^{(c)}(\mathbf{X}, \mathbf{Y}^{(n)} | \ell, \omega(n, \cdot)) = d_{\mathbb{T}}^{(c)}(\mathbf{X}_\ell, \mathbf{Y}_{\omega(n, \ell)}^{(n)}) = d_{\mathbb{T}}^{(c)}(\tau_\ell, \chi_{\omega(n, \ell)}^{(n)}) > 0$ only when $\varepsilon_\ell(\mathbf{X}) = 1$ and $\varepsilon_{\omega(n, \ell)}(\mathbf{Y}^{(n)}) = 1$. Additionally, if $\mathcal{R}(\eta, \tau) = \mathcal{R}(\eta', \tau')$, then $\|\eta\|_1 = \|\eta'\|_1$ and $\{\tau_\ell | \eta_\ell = 1, \ell = 1 : L\} = \{\tau'_\ell | \eta'_\ell = 1, \ell = 1 : L\}$, which implies $Q^{(r)}(\eta, \tau) = Q^{(r)}(\eta', \tau')$. Since $\mathcal{R} \circ \mathcal{A} = \mathbf{I}$, by the virtue of Lemma 1, solving (17) with the distance form given by (19) amounts to solving (23) and applying the transformation \mathcal{R} to recover the optimal multi-object trajectory.

H. Proof of Proposition 6

Using the definition of $Q_\omega^{(r)}(\eta, \tau)$ in (20),

$$\begin{aligned} Q_\omega^{(r)}(\eta, \tau^*(\eta, \omega)) &= \min_{\tau \in \mathbb{T}^L} \sum_{\ell=1}^L \eta_\ell \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell) + (1 - \eta_\ell) \bar{\phi}_{\omega, \ell}^{(r)}(\eta) \\ &= \sum_{\ell=1}^L \min_{\tau_\ell \in \mathbb{T}} \eta_\ell \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell) + \sum_{\ell=1}^L (1 - \eta_\ell) \bar{\phi}_{\omega, \ell}^{(r)}(\eta). \end{aligned}$$

For each ℓ , $\tau_\ell^*(\eta, \omega) = \arg \min_{\tau_\ell \in \mathbb{T}} \eta_\ell \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell)$. If $\eta_\ell = 0$ or $\xi_{\omega(n, \ell)}^{(n)} = 0$ for all $n = 1 : N$, any value of $\tau_\ell^*(\eta, \omega)$ yields the same cost. Hence, $\tau_\ell^*(\eta, \omega) = \mathbf{0}$ is an optimal solution. If $\eta_\ell = 1$, then $\eta_\ell \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell) = \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell)$, and

$$\begin{aligned} \tau_\ell^*(\eta, \omega) &= \arg \min_{\tau_\ell \in \mathbb{T}} \phi_{\omega, \ell}^{(r)}(\eta, \tau_\ell) \\ &= \arg \min_{\tau_\ell \in \mathbb{T}} \sum_{n=1}^N \xi_{\omega(n, \ell)}^{(n)} \frac{d_{\mathbb{T}}^{(c)}(\tau_\ell, \chi_{\omega(n, \ell)}^{(n)})^r}{\kappa(\|\eta\|_1, \|\xi^{(n)}\|_1)}. \end{aligned}$$

II. EXTENSIONS ON CONSENSUS VIA GIBBS SAMPLING

This section provides a mathematical proof for the convergence of the Gibbs sampler proposed in Subsection V-B, and discusses efficient implementation strategies.

A. Convergence Properties

It is not obvious that the Gibbs sampler in Algorithm 7 can visit all possible states in a finite number of steps. However, this is demonstrated in the following proposition.

Proposition A1. Starting from any valid $(\eta_{1:L}, \omega^{(1:N)})$ the Gibbs sampler defined by the conditionals $\rho_\ell(\cdot | \eta_{1:\ell-1}, \eta_{\ell+1:L}, \omega^{(1:N)})$ and $\rho_{L+n}(\cdot | \hat{\eta}_{1:L}, \hat{\omega}^{(1:n-1)}, \omega^{(n+1:N)})$ converges to the distribution $\rho(\eta_{1:L}, \omega^{(1:N)})$ at an exponential rate.

Proof. We need to show that the chain can move from any valid state $(\eta_{1:L}, \omega^{(1:N)})$ to any other valid state $(\hat{\eta}_{1:L}, \hat{\omega}^{(1:N)})$ in a finite number of steps. Noting that $\rho(\eta_{1:L}, \omega^{(1:N)})$ is zero for invalid samples, i.e., when $\omega^{(n)} \notin \Omega^{(n)}(\eta_{1:L})$ for any $n \in \{1 : N\}$, and is positive otherwise. Thus, we can transit from any valid $(\eta_{1:L}, \omega^{(1:N)})$ to any other valid $(\hat{\eta}_{1:L}, \hat{\omega}^{(1:N)})$ in a finite number of steps. We also demonstrate that it is possible to move from $\eta_{1:L}$ to $\hat{\eta}_{1:L}$ in a finite number of steps. For any $\ell \in \{1 : L\}$ and $n \in \{1 : N\}$, we proceed with the following cases:

Case 1: Suppose $\|\eta_{1:L}\|_1 = \|\xi^{(n)}\|_1$. Changing η_ℓ from 0 to 1 (or vice versa) to reach some intermediate state $\hat{\eta}_{1:L}$ will always satisfy $\omega^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$. This permits a transition from $(\eta_{1:L}, \omega^{(1:N)})$ to the intermediate state $(\hat{\eta}_{1:L}, \omega^{(1:N)})$ and then $(\hat{\eta}_{1:L}, \hat{\omega}^{(1:N)})$.

Case 2: Suppose $\|\eta_{1:L}\|_1 < \|\xi^{(n)}\|_1$. Changing η_ℓ from 1 to 0 to reach an intermediate state $\hat{\eta}_{1:L}$ will satisfy $\omega^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$, allowing a transition from $(\eta_{1:L}, \omega^{(1:N)})$ to $(\hat{\eta}_{1:L}, \omega^{(1:N)})$. If $\xi_\ell^{(n)} = 1$, changing η_ℓ from 0 to 1 to reach $\hat{\eta}_{1:L}$ will also satisfy $\omega^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$. However, if $\xi_\ell^{(n)} = 0$, to change η_ℓ from 0 to 1 we first need to change $\omega^{(n)}$ to $\hat{\omega}^{(n)}$ as follows. For some $\hat{\ell}$ where $\eta_{\hat{\ell}} = 0$ and $\xi_{\hat{\ell}}^{(n)} = 1$, we select some $\hat{\omega}^{(n)} \in \Omega^{(n)}(\eta_{1:L})$ such that: $\hat{\omega}_{\hat{\ell}}^{(n)} = \omega_{\hat{\ell}}^{(n)}$; $\hat{\omega}_{\hat{\ell}}^{(n)} = \omega_{\hat{\ell}}^{(n)}$;

and $\dot{\omega}_{\check{\ell}}^{(n)} = \omega_{\check{\ell}}^{(n)}$ for $\check{\ell} \neq \ell, \hat{\ell}$. Since $\|\eta_{1:L}\|_1 < \|\xi^{(n)}\|_1$, there always exists a $\check{\ell}$ that satisfies our requirement. Since $\dot{\omega}^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$, it allows us to move from $(\eta_{1:L}, \omega^{(1:N)})$ to $(\eta_{1:L}, \dot{\omega}^{(1:N)})$ to $(\hat{\eta}_{1:L}, \dot{\omega}^{(1:N)})$ and then $(\hat{\eta}_{1:L}, \dot{\omega}^{(1:N)})$.

Case 3: Suppose $\|\eta_{1:L}\|_1 > \|\xi^{(n)}\|_1$. Changing η_ℓ from 0 to 1 to reach an intermediate state $\hat{\eta}_{1:L}$ will satisfy $\omega^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$, allowing a transition from $(\eta_{1:L}, \omega^{(1:N)})$ to $(\hat{\eta}_{1:L}, \omega^{(1:N)})$. If $\xi_\ell^{(n)} = 0$, changing η_ℓ from 1 to 0 to reach $\hat{\eta}_{1:L}$ will also satisfy $\omega^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$. However, if $\xi_\ell^{(n)} = 1$, to change η_ℓ from 1 to 0 we first need to change $\omega^{(n)}$ to $\dot{\omega}^{(n)}$ as follows. For some $\check{\ell}$ where $\eta_{\check{\ell}} = 1$ and $\xi_{\check{\ell}}^{(n)} = 0$, we select some $\dot{\omega}^{(n)} \in \Omega^{(n)}(\eta_{1:L})$ such that: $\dot{\omega}_{\check{\ell}}^{(n)} = \omega_{\check{\ell}}^{(n)}$; $\dot{\omega}_{\hat{\ell}}^{(n)} = \omega_{\hat{\ell}}^{(n)}$; and $\dot{\omega}_{\check{\ell}}^{(n)} = \omega_{\check{\ell}}^{(n)}$ for $\check{\ell} \neq \ell, \hat{\ell}$. Since $\|\eta_{1:L}\|_1 > \|\xi^{(n)}\|_1$, there always exists a $\check{\ell}$ that satisfies our requirement. Since $\dot{\omega}^{(n)} \in \Omega^{(n)}(\hat{\eta}_{1:L})$, it allows us to move from $(\eta_{1:L}, \omega^{(1:N)})$ to $(\eta_{1:L}, \dot{\omega}^{(1:N)})$ to $(\hat{\eta}_{1:L}, \dot{\omega}^{(1:N)})$ and then $(\hat{\eta}_{1:L}, \dot{\omega}^{(1:N)})$.

This implies the Markov chain is irreducible and regular, hence the convergence to the target distribution $\rho(\eta_{1:L}, \omega^{(1:N)})$ at an exponential rate, following from Lemma 4.3.4 of [81]. ■

B. Enhancement Strategies

In this section, we introduce two techniques to respectively improve the convergence rate and the efficiency of the Gibbs sampler proposed in Subsection V-B.

1) Rearranging Non-Existence Assignment: For an existence array $\eta_{1:L}$ and some existence assignments $\omega^{(1:N)}, \dot{\omega}^{(1:N)} \in \Omega^{(1:N)}(\eta_{1:L})$, if $\omega_\ell^{(n)} = \dot{\omega}_\ell^{(n)}$ for all ℓ that $\eta_\ell = 1$ then $S^{(r)}(\eta_{1:L}, \omega^{(1:N)}) = S^{(r)}(\eta_{1:L}, \dot{\omega}^{(1:N)})$, i.e., $\omega^{(1:N)}$ and $\dot{\omega}^{(1:N)}$ yield the same cost value if they only differ at the components where $\eta_\ell = 0$ (assignment of the non-existence elements). If $\eta_\ell = 0$ and the pair-wise distances among $\chi_{\omega_\ell^{(1)}}^{(n)}, \dots, \chi_{\omega_\ell^{(N)}}^{(n)}$ are lower than ones among $\chi_{\omega_\ell^{(1)}}^{(n)}, \dots, \chi_{\omega_\ell^{(N)}}^{(n)}$, replacing $\omega^{(1:N)}$ with $\dot{\omega}^{(1:N)}$ could increase the probability of η_ℓ transiting from 0 to 1 because their distances to the mean would be shorter.

For some $\eta_{1:L}$, $\bar{L} = \sum_{\ell=1}^L (1 - \eta_\ell)$, we define the mapping $\bar{\mathcal{E}} : \Pi_L \rightarrow \mathcal{F}_{\bar{L}}(\{1 : L\})$ (where $\mathcal{F}_{\bar{L}}(\{1 : L\})$ is all finite subsets of $\{1 : L\}$ with \bar{L} elements) such that $\bar{\mathcal{E}}(\omega) = \{\omega_\ell | \eta_\ell = 0, \ell = 1 : L\}$. For some $\omega \in \Pi_L$, we define the mapping $\mathcal{E}_\omega : \mathcal{F}_{\bar{L}}(\{1 : L\}) \rightarrow \Pi_L$ such that if $\mathcal{E}_\omega(\{\bar{\omega}_{1:\bar{L}}\}) = \dot{\omega}$ then

$$\begin{cases} \dot{\omega}_\ell = \omega_\ell, & \text{if } \eta_\ell = 1 \\ \dot{\omega}_\ell = \bar{\omega}_m, m = \sum_{i=1:\ell} (1 - \eta_i), & \text{if } \eta_\ell = 0 \end{cases}.$$

We note that if $\bar{\omega} \in \Pi_{\bar{\mathcal{E}}(\omega)}$ is a permutation of $\bar{\mathcal{E}}(\omega)$ and $\omega \in \Omega^{(n)}(\eta_{1:L})$ then $\mathcal{E}_\omega(\bar{\omega}) \in \Omega^{(n)}(\eta_{1:L})$. Thus, we can update $\omega^{(n)} \in \Omega^{(n)}(\eta_{1:L})$ to $\dot{\omega}^{(n)} \in \Omega^{(n)}(\eta_{1:L})$ using some $\bar{\omega}^{(n)} \in \Pi_{\bar{\mathcal{E}}(\omega^{(n)})}$ such that $\dot{\omega}_\ell^{(n)} = \mathcal{E}_{\omega^{(n)}}(\bar{\omega}^{(n)})$. Noting that $S^{(r)}(\eta_{1:L}, \omega^{(1:N)}) = S^{(r)}(\eta_{1:L}, \dot{\omega}^{(1:N)})$. To achieve low pair-wise distances among $\chi_{\dot{\omega}_\ell^{(1)}}^{(n)}, \dots, \chi_{\dot{\omega}_\ell^{(N)}}^{(n)}$, we sample $\bar{\omega}^{(1:N)}$ from the distribution

$$\bar{\rho}(\bar{\omega}^{(1:N)}) \propto e^{-\alpha(R^{(r)}(\bar{\omega}^{(1:N)}))} \prod_{n=1}^N \mathbf{1}_{\Pi_{\bar{\mathcal{E}}(\omega^{(n)})}}(\bar{\omega}^{(n)}),$$

where

$$\begin{aligned} R^{(r)}(\bar{\omega}^{(1:N)}) &= \sum_{\ell=1}^{\bar{L}} \zeta_\ell^{(r)}(\bar{\omega}^{(1:N)}), \\ \zeta_\ell^{(r)}(\bar{\omega}^{(1:N)}) &= \sum_{n=1}^N \sum_{j=1, j \neq n}^N \xi_{\bar{\omega}_\ell^{(n)}}^{(r)} \xi_{\bar{\omega}_\ell^{(j)}}^{(j)} d_{\mathbb{T}}^{(c)}(\chi_{\bar{\omega}_\ell^{(n)}}^{(n)}, \chi_{\bar{\omega}_\ell^{(j)}}^{(j)})^r \\ &\quad + \xi_{\bar{\omega}_\ell^{(n)}}^{(n)} (1 - \xi_{\bar{\omega}_\ell^{(j)}}^{(j)}) p^r + (1 - \xi_{\bar{\omega}_\ell^{(n)}}^{(n)}) \xi_{\bar{\omega}_\ell^{(j)}}^{(j)} p^r, \end{aligned}$$

Sampling from $\bar{\rho}$ can be done using the Gibbs sampling technique proposed in Subsection V-B of the main text.

2) Efficient Proposal Distribution: We observe that if the trajectories in $\mathbf{Y}^{(1:N)}$ are assigned to the same trajectory \mathbf{X}_ℓ in the mean multi-object trajectory, they are expected to be close to each other. Otherwise, if $\mathbf{Y}^{(n)}$ has no trajectories assigned to \mathbf{X}_ℓ , it contributes p^r to the cost function. Therefore, we propose the following proposal conditional $\tilde{\rho}_{L+n}$ in place of ρ_{L+n} in Subsection V-B of the main text to sample for the existence assignment, i.e.,

$$\begin{aligned} \tilde{\rho}_{L+n}(\omega^{(n)} | \hat{\eta}_{1:L}, \dot{\omega}^{(1:n-1)}, \omega^{(n+1:N)}) &= \\ e^{-\alpha(\tilde{S}^{(r,n)}(\hat{\eta}_{1:L}, [\dot{\omega}^{(1:n-1)}, \omega^{(n:N)}]))} \mathbf{1}_{\Omega^{(n)}(\eta_{1:L})}(\omega^{(n)}), \end{aligned}$$

where

$$\begin{aligned} \tilde{S}^{(r,n)}(\eta, \omega) &= \sum_{\ell=1}^L \eta_\ell \phi_{\omega, \ell}^{(r,n)}(\eta) + (1 - \eta_\ell) \dot{\phi}_{\omega, \ell}^{(r,n)}, \\ \phi_{\omega, \ell}^{(r,n)}(\eta) &= \sum_{j=1, j \neq n}^N \xi_{\omega_\ell^{(n)}}^{(n)} \xi_{\omega_\ell^{(j)}}^{(j)} [d_{\mathbb{T}}^{(c)}(\chi_{\omega_\ell^{(n)}}^{(n)}, \chi_{\omega_\ell^{(j)}}^{(j)})]^r \\ &\quad + \xi_{\omega_\ell^{(n)}}^{(n)} (1 - \xi_{\omega_\ell^{(j)}}^{(j)}) p^r + (1 - \xi_{\omega_\ell^{(n)}}^{(n)}) \xi_{\omega_\ell^{(j)}}^{(j)} p^r, \\ \dot{\phi}_{\omega, \ell}^{(r,n)} &= \sum_{j=1, j \neq n}^N \xi_{\omega_\ell^{(j)}}^{(j)} p^r. \end{aligned}$$

Compared to ρ_{L+n} , we do not need to compute the mean trajectory τ_ℓ^* when computing $\tilde{\rho}_{L+n}$. Thus, the complexity of Algorithm 7 reduces to $\mathcal{O}(G_m L^2 Z)$.